A FRAMEWORK FOR THE DESIGN OF ADAPTIVE CURRICULUM FOCUSED ON THE DEVELOPMENT OF MATHEMATICAL THINKING

Doctoral Program in Mathematical Education

Thesis presented as an option toward the scientific degree of

Doctor in Mathematics Education

Mg. Carlos Alberto Díez Fonnegra

Universidad Antonio Nariño

Bogotá D.C., Colombia

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ABSTRACT

The objective of this research was to design a methodology for the design of adaptive curricula that foster the development of mathematical thinking, considering the students' learning conditions. In this work, the learning conditions related to the modes of thinking proposed by Sierpinska (2000) were especially considered, since these modes are related to linear algebra, which was the field of mathematics where this research was applied.

For this purpose, the design-based research method was used, within the qualitative research approach. Following this method, a test of characterization of modes of thinking (geometric and arithmetic) was elaborated and an adaptive system for learning management (ALMS) was built, of which two iterations were carried out in its implementation with students of courses related to linear algebra.

From this, it was possible to identify essential characteristics that adaptive curricula should contain to foster the development of mathematical thinking, thus overcoming the current state in the design of this type of curricula, which is mainly focused on learning results and not on the thinking processes involved.

SÍNTESIS

Esta investigación tuvo como objetivo diseñar una metodología para el diseño de currículos adaptativos que favorezcan el desarrollo del pensamiento matemático atendiendo las condiciones de aprendizaje de los estudiantes. En este trabajo se consideraron especialmente las condiciones de aprendizaje relacionadas con los modos de pensamiento propuestos por Sierpinska (2000), dado que dichos modos están relacionados con el álgebra lineal, que fue el campo de las matemáticas donde se aplicó esta investigación.

Para ello, se utilizó el método de investigación basada en diseño, dentro del enfoque cualitativo de investigación. Siguiendo este método, se elaboró una prueba de caracterización de modos de pensamiento (geométrico y aritmético) y se construyó un sistema adaptable para la gestión del aprendizaje (ALMS, por sus siglas en inglés), del que se realizaron dos iteracciones en su implementación con estudiantes de cursos relacionados con el álgebra lineal.

A partir de esto, se lograron identificar características esenciales que deben contener los currículos adaptativos para favorecer el desarrollo del pensamiento matemático, con lo que se supera el estado actual en el diseño de este tipo de currículos, centrado fundamentalmente en los resultados del aprendizaje y no en los procesos de pensamiento involucrados.

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INTRODUCTION

The words or the language, as they are written or spoken, do not seem to play any role in my mechanism of thought. The psychical entities which seem to serve as elements in thought are certain signs and more or less clear images which can be "voluntarily" reproduced or combined ... The above-mentioned elements are, in my case, of visual and some of muscular type. Conventional words or other signs have to be sought for laboriously only in a secondary stage.

Albert Einstein (1954)

The educational system must be adjusted to the needs and aspirations of individuals within a given society at a given time. This statement, which sounds tautologically true, has not been the prevailing reality in history. In different periods, it has been possible to see that education frequently falls behind in meeting the social, economic, political, technological, and other needs of society (Commission européenne & High Level Group on the modernisation of higher education, 2014). This happens for various reasons ranging from political, economic, and philosophical paradigms to technological and logistical difficulties.

Today's society, framed in the global paradigm and with the pressure of the technological revolution 4.0, is no exception to this situation. Some characteristics of this revolution (Schwab, K. 2017), such as:

- Emergence and relevance of knowledge-based activities.
- Incorporation of ICTs in most activities.
- Preponderance of mathematics and its applications.
- High speed in the evolution of technology.

have led to an increase in the study of areas related to mathematics and the inclusion of mathematics in areas in which it was not previously prominent. This has caused more and more academic programs at the university level to have some component of formation in mathematics, and, due to the economic conditions of educational institutions, the mathematical components of various of these programs are commonly taught to diverse groups of students under a one-size-fits-all model.

The situation mentioned above, i.e., the implementation of a homogeneous education oriented to students with heterogeneous characteristics, is not an exclusive problem of the university educational environment, but also happens at the level of basic training. However, in the latter environment there are other conditions that need to be studied in combination with this problem, such as the choice between general or specific education in the high school, or whether this level of education should be oriented towards providing basic content or forming general processes, etc., so the present doctoral thesis restricts its field of action for the study of the problem exclusively to the university environment.

The postgraduate degree programs in Mathematics Education of the Universidad Antonio Nariño have been aware of and considered this need, and therefore, one of its lines of research is the generation of a more challenging mathematics curriculum for all students consistent with the needs of the twenty-first century, line in which this doctoral thesis is framed.

A possible cause for the lack of efficiency in the education of students is the homogeneous education model. Specifically, regarding mathematics, the homogeneous education model has brought about related difficulties, such as students not seeing the relevance of the proposed mathematical formation for their careers, which causes them not to have adequate engagement with their mathematics courses and consequently it is difficult for them to develop their mathematical thinking. This model has also caused a high dropout rate in mathematics courses and, therefore, in university programs. At this point, the situation is so critical that it is thought to be normal that mathematics courses have failure levels higher than 30% or 40%, and that students abandon their university studies because they were not able to overcome the 'filter' of mathematics courses.

This situation has raised serious doubts about the role of university education, not only in mathematics, which has led to the formulation of analytical documents on the five continents (Pedroza Flores, 2018).

In the academic community in general attention to this need has led in the past 20 years to an intense level of proposal of models of adaptive education of multiple types, with the aim of evolving from education for all people to education for each person. *"One aspect that stands out in these academic innovations is the redefinition of teaching-learning because it disrupts several points in the model, as well as in university academic processes and practices. The trend in teaching and learning is towards an adaptive model. Teachers are being renewed in the face of the increasingly assimilated use of mixed intelligence in the classroom and outside it. Many examples of this are underway"*[1].

These models have ranged from the most traditional (where the teacher introduces the adaptation himself), to the most advanced (using state-of-the-art technology, such as artificial intelligence and data analytics).

Such is the level of inclusion of adaptation in education that there are many patents oriented in this direction (O'Dwyer, B., & Krishnan, R., 2018) (Dohring, D. C. et al, 2018) (Hibbs, A. D., et al, 2016).

Along with these proposals, numerous studies have been carried out to assess the effectiveness of these models, with very different results that do not point to clearly identifiable models that should preferentially be put in practice. According to this research, it is not yet clear which are the

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[1] Pedroza Flores, R. (2018). La universidad 4.0 con currículo inteligente 1.0 en la cuarta revolución industrial. *Revista Iberoamericana para la Investigación y el Desarrollo Educativo, 9*(17), p. 15.

determining factors that must be adapted, nor which are the best systems for adapting them, so that adaptive education has not always been a successful alternative (Kirschner, P. A., 2017).

Another problematic situation lies in the fact that most models of adaptive education have been proposed not by educators, not even by specialists in specific areas, but by engineers in fields related to systems and programming who, although they may have some specialization in their studies related to education, do not have their main interest centered on educational issues. This has meant that these models are often oriented towards the technical characteristics of the system and even the psychological characteristics of the students, but not so much towards the specific characteristics of the teaching content, especially if one thinks of the content not as subjects, but as thinking processes. Thus, most adaptation models do not distinguish between two students with different ways of thinking when both obtain the same result, i.e., the system will propose the same learning path for students who obtain equivalent results at different moments of measurement, even if they have arrived at these results by completely opposite or at least very different ways of thinking.

For the same reason, that is, that those who have developed the models are not educators, basic and current pedagogical concepts such as curriculum design, formative evaluation, development of thinking, among others, and much less modern pedagogical theories, have not been incorporated into these models.

Another difficulty of most adaptive education proposals is that they do not consider the relationship between students and teacher nor students with one another. And yet, the fact that there is a need to adapt teaching-learning processes to the diverse characteristics of students is not a determinant for not being able to use this same diversity in the relationships among these students to enhance learning outcomes.

Finally, models of adaptive education are marginal and, due to technical, political and economic factors, have not been widely applied in educational institutions, whether dedicated to basic or university education (Commission européenne & High Level Group on the modernisation of higher education, 2014).

Based on this characterization of the problematic of the situation of adaptive education at present, the research proposed in this doctoral work, whose central problem is how the development of mathematical thinking can be fostered by considering the individual learning conditions of students, finds its justification.

It is anticipated that a solution to this problem can be found by proposing, as a theoretical contribution, the development of a framework for the design of adaptive curricula aimed at the development of mathematical thinking. And, to validate this proposed framework, and as a practical contribution of this research, the framework proposed in the design of an adaptive curriculum for the development of mathematical thinking related to linear algebra will be developed and used.

With this research, which aims at adaptive education for the teaching-learning of mathematics, the following objectives are intended to be achieved:

Overarching objective

A methodology for the design of adaptive curricula that fosters the development of mathematical thinking, considering the learning conditions of the students.

Specific objectives

1. Determination of the learning conditions from which the curriculum can be adapted.
2. Determination of the design characteristics of an adaptive curriculum for the development of mathematical thinking.

3. Proposal of a framework for the design of an adaptive curriculum oriented to the development of different forms of mathematical thinking.
4. Design of a curriculum for teaching linear algebra based on the above framework.

The field of action of this research is restricted to the study of methodologies for the development of mathematical thinking through the design and implementation of adaptive curricula in the university environment.

For the doctoral research proposed, four chapters are presented below. The first chapter recounts the state of the art in the different dimensions of the research: adaptive education, including learning conditions and adaptive education systems; curriculum design and evaluation; development of mathematical thinking; and teaching linear algebra.

The second chapter explores various theories that will be necessary for the formulation of the framework. These theories are examined and structured in the same dimensions mentioned in the state of the art.

The third chapter describes the research methodology, framed in the design-based research method, and presents the way in which the two iterations that were carried out in the doctoral research were conducted.

The fourth chapter analyzes the results obtained in the implementation of the research, leading to the proposal of a methodology for the design of adaptive curricula that foster mathematical thinking.

The document also contains four appendices with relevant information for a full understanding of the ideas presented in the thesis.

CHAPTER 1. STATE OF THE ART

This section presents reviews of some of the most recent and high impact articles in each of the conceptual categories underlying this work. The organization of each part of this section will be done in chronological order, starting with the oldest articles, unless otherwise stated.

1.1 Adaptive education

The concept of adaptive education has taken on great relevance in recent years. This is due to needs and possibilities: the need for having people learn more efficiently and effectively and the possibilities of using computer systems that make adaptation to multiple learning styles feasible.

This section presents recent studies related to adaptive education and its main concepts: adaptive learning and teaching, learning styles, and adaptive learning environments.

1.1.1 Adaptive learning and teaching

This section reviews research that describes the characteristics and principles of adaptive learning and teaching.

1.1.1.1 Ontologies for personalized adaptive learning[2]

This article proposes an ontological approach to designing a personalized e-learning system which creates contents tailored for individual learners. To that end, it defines the concept of ontology as a formal and explicit specification of a conceptualization. According to Gruber (1993), thus, *"ontology*

[2] Yarandi, M. et al (2012). Ontologies for personalised adaptive learning. In *Advances in Computing & Technology*. University of East London, School of Architecture Computing and Engineering.

represents the conceptual explanation of the specific content, as it helps to identify the appropriate elements and relationships in a given set of knowledge domains".[3]

The authors present the difference between the two most important types of adaptive education systems: intelligent tutoring systems, which adapt content to the learner but within certain borders; and adaptive hypermedia systems, which provide content and navigation paths that adapt to the user's needs. Based on these definitions, they give an account of some of the systems of adaptive education that have existed so far.

Based on Brusilovsky (2001), the adaptation variables are determined according to the characteristics of the learners: learning style, level of knowledge, ability of the learner and preferences, and for each one they propose a dichotomous categorization of values.

In principle, for the present thesis, these are adequate variables to make the process adaptive. However, in this research the intention is to explore other variables of adaptation more focused on the learning process and not only on the characteristics of the student.

The innovative aspect of this article is the proposal of three ontological models (concept approached from the computational point of view): the student model, the domain model and the content model based on the research of Yarandi et al (2011).

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3 Yarandi, M. et al (2012). Ontologies for personalised adaptive learning. In Advances in Computing & Technology. University of East London, School of Architecture Computing and Engineering. p. 1.

1.1.1.2 Use of adaptive study material in education in an e-learning environment[4]

The article presents a study of how materials should be structured in adaptive educational environments. For this, it proposes that an adaptive system should be composed of three basic modules: the student module, the study material module, and the administration module.

On the other hand, the authors rely on Gagné (1975), stating that the educational process should have the following steps: gaining attention, formulating goals, recognizing prior knowledge, presenting new topics, guiding students, giving feedback, evaluating performance, and making sure they remember what they have learned. This way of modeling this process is based on the conception that the student should learn things, but not on the conception that the student should develop his thinking. Therefore, the present research will propose another way to model the educational process.

The structure proposed by the authors also reflects the way in which they construct the materials, since they divide them into frames that have contents adapted to different learning styles and different degrees of depth which allows them to represent graphically the student's progress through the different topics of content. Although this research work will be based on the development of thinking and not only on the learning of content, the articulation of these two variables could be favorable for the design of the materials here.

1.1.1.3 Personalized learning: One-Size-Fits-One model[5]

This article shows how the failure of the One-Size-Fits-All model gives rise to the need for personalized learning as a viable alternative to address the specific needs of each learner.

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4 Kostolányová, K., & Šarmanová, J. (2014). Use of Adaptive Study Material in Education in E-Learning Environment. Electronic Journal of e-Learning, 12(2), 172-182.
5 Bettahi, J. (2018). Personalized Learning: One-Size-Fits-One Model.

It also discusses the definition of the concept of personalized learning from different perspectives, one in which the same curriculum is intended to reach the greatest number of people, another in which groups of people with similar characteristics are thought of, and finally, one in which the needs and possibilities of individuals are sought to be satisfied.

Based on this discussion, the author assumes a point of view that gives a multi-dimensional definition of personalized learning, based on needs and learning abilities, interests and learning goals, and learning preferences. A fundamental part of the present research has to do with defining the categories of curriculum adaptation, and this article provides very appropriate elements in this regard.

The article also describes the potential of technology for supporting personalized learning and shows the different technological categories available for this purpose: learning repositories, tutoring platforms, learning management systems and adaptive web-based educational systems.

Finally, Bettahi (2018) presents a framework for a web-based platform to enhance learners' personalized learning experiences, called PLEP (Personalized Learning Educational Platform).

1.1.1.4 Personalización y adaptación en un ambiente virtual de aprendizaje basada en estilos, conocimiento previo y errores frecuentes[6]

In this article, the authors propose that the adaptation of a learning system can be done according to various characteristics of the students: knowledge levels, cognitive skills, learning styles, emotions, reactions, frequent errors, among others.

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[6] Gonzalez, M. P. et al (2019). Personalización y adaptación en un ambiente virtual de aprendizaje basada en estilos, conocimiento previo y errores frecuentes. In *XXI Workshop de Investigadores en Ciencias de la Computación (WICC 2019, Universidad Nacional de San Juan)*.

Among all of them, they choose to use the predominant learning styles, the previous knowledge of the students and the analysis of frequent errors. They introduce this last factor as an advance made in previous work developed by the research group to which they belong.

For the characterization of learning styles, they use the model designed by Felder and Silverman (1988), the Index of Learning Styles.

This model, together with the analysis of previous knowledge and the frequent errors were implemented in the Learning Management System (LMS) Moodle.

1.1.1.5 Trends and development in technology-enhanced adaptive/ personalized learning: A systematic review of journal publications from 2007 to 2017[7]

This article reviews work between 2007 and 2017 related to adaptive/personalized learning. The presentation begins by relating learning methods and theories that reinforce the idea that learning is a personal and differentiated process. In this sense, the authors highlight, among other things, that the United States National Education Technology Plan 2017 states that personalized learning is defined as *"instruction in which the pace of learning and the instructional approach are optimized for the needs of each learner. Learning objectives, instructional approaches, and instructional content (and its sequencing) may all vary based on learner needs. In addition, learning activities are meaningful and relevant to learners, driven by their interests, and often self-initiated"*.[8]

They also make a difference between the terms "adaptive" and "personalized", although they recognize that in the studies reviewed, these are terms that are often freely interchanged. Adaptive

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[7] Xie, H. et al (2019). Trends and development in technology-enhanced adaptive/personalized learning: A systematic review of journal publications from 2007 to 2017. *Computers & Education*, 103599.

[8] U.S. Department of Education, Office of Educational Technology. (2018, Abril 14). Reimagining the role of technology in education: 2017 national education technology plan update. Retrieved from https://tech.ed.gov/files/2017/01/NETP17.pdf. p. 2.

learning refers to the active process of modifying teaching conditions based on student characteristics, while personalized learning refers to differentiated modes that are proposed from the beginning of the learning process.

The article highlights the importance of the role of technology in adaptive learning, as evidenced by the number of articles that were published in the period under analysis in various journals.

The authors review articles that have had a high impact as measured by citations (using Google Scholar as a reference). However, none of these studies is related to the learning of mathematics.

This systematic review of adaptive/personalized learning seeks to identify development, trends, challenges, and potential research directions.

As a result, the authors point out that most of the studies carried out have been aimed at high school students, especially in subjects related to computing, science, and language, with a few devoted to mathematics.

They also show that the greatest tendency is in the personalization of contents, followed by the personalization of learning paths and user interfaces, developed mainly in traditional computers.

The two most relevant learning outcomes are those related to cognition and affection. In this sense, it is worth highlighting how little the projects have referred to other learning outcomes such as skills and thinking processes. These last two learning outcomes will be central axes of the present doctoral research.

In this same sense, they emphasize that research is mainly directed at measuring learning outcomes, but not at the learning process. This will be another distinguishing feature of this doctoral research.

This review also discusses new trends in automatic extraction of the structure of learning content through machine learning.

The authors conclude their work by showing the importance that adaptive/personalized learning has gained in recent times within the educational context and by exposing the challenges presented for its implementation on a large scale.

1.1.1.6 Pedagogical guidelines for the creation of adaptive digital educational resources: a review of the literature[9]

In this article, the authors, based on a systematic review of the literature related to the topic of adaptive learning, propose a user model, that contemplates experiences, interests, and tastes, in order to improve student learning.

Initially, they highlight an observation, which is also an issue that will be dealt with in this doctoral thesis, that is related to the difference in approaches that the topic of adaptive education has when it is dealt with in computer journals, engineering, and educational technology, in contrast to the way it is dealt with in education journals.

This difference in approach is first seen in the number of articles and projects reported: while articles are abundant in the first type of journals, in the second type of journal there are few results.

They emphasize that in the first type of articles, i.e., those published in computer, engineering and educational technology journals, important advances in adaptation are noted with the use, for example, of artificial intelligence, while in articles published in educational journals, the use of better technological resources is needed to achieve better adaptation.

[9] Rozo, H., & Real, M. (2019). Pedagogical Guidelines for the Creation of Adaptive Digital Educational Resources: A Review of the Literature. *Journal of Technology and Science Education*, *9*(3), 308-325.

Because of the rigorous selection methodology of the texts used for this systematic study, the authors propose results in the following categories: digital educational resources, learning styles, adaptiveness, and user model.

In relation to digital educational resources, they present diverse articles ranging from the most traditional: resource objectives, activities, instructional design, and feedback, in Pérez et al (2001), to research that takes more into account technical and usability characteristics such as self-contained design, reutilization, and small units, (Gascueña et al, 2005).

With respect to learning styles, they gather various definitions, formulating two fundamental scenarios: one in which learning styles are defined as the ability of everyone to process and understand information; and the second that focuses on the different strategies and ways of organizing information and content for later analysis. They propose that these two scenarios are part of the same process that is learning, and they show how various proposals integrate these two scenarios in different ways.

From the point of view of the present doctoral research, these two scenarios are insufficient to describe learning styles. A fundamental scenario is needed, that of the thinking processes that are put into play when learning.

For the purpose of adaptation, the authors give a description of various criteria for carrying out this process, for example: the presentation of the information and contents, the context of use, the amount of help to the learner, the strategy and the narrative. They emphasize differentiating between adaptable systems, which are those that allow the user to modify the characteristics of the system depending on his preferences, and adaptive systems, which are those that automatically adapt to the user, according to his needs, from assumptions made by the system.

In this same field, they refer to Brusilovsky & Maybury (2002), who establish that in the adaptation there are three dynamic elements: contents, navigation, and presentation.

Finally, the article refers to the concept of the user model. This model interacts with the content model and with the adaptation characteristics to generate an adaptive system.

This article, written by Colombian researchers, presents very valuable elements for the present doctoral thesis, because apart from showing a systematic review of the state of the art, it addresses conceptually important elements to achieve the purpose of the present research.

1.1.1.7 Implementing Adaptive e-Learning Conceptual Model: A Survey and Comparison with Open Source LMS[10]

This article shows an analysis of various LMSs and, based on this, proposes the design of a particular one that fits the needs of the teaching and learning process.

This design is based on applied surveys, both to teachers and students, about the functionalities that can be useful to them in an LMS.

This article is useful to identify the components of a conceptual model for an adaptive e-learning system and to develop an LMS suitable for the real application of the curriculum.

1.1.2 Learning styles, an example of learning conditions

Learning styles are one of the main ways in which education systems adapt. Moreover, in a broad way one could consider that modes of thinking, the learning condition that will be used to make the adaptation in this thesis, are in some way learning styles.

[10] Alameen, A., & Dhupia, B. (2019). Implementing Adaptive e-Learning Conceptual Model: A Survey and Comparison with Open Source LMS. *International Journal of Emerging Technologies in Learning (iJET)*, *14*(21), 28-45.

In the following material, some of the articles reviewed are related to measuring the effectiveness of detecting the learning styles of students, whereas others show descriptions of the most recent ways to detect learning styles and, finally, there are two articles that review the state of the art in this regard.

1.1.2.1 Adaptivity in e-learning systems[11]

In this short but descriptive article, the authors present the fundamental elements of adaptive learning. First, they set out the factors that determine students' motivation: capturing their attention, engagement, interactivity, and the way content is structured. They also present a proposal for a student model, which has four elements: learning style, expectations, motivation, and level of knowledge. With respect to learning styles, they mention the classification of Felder & Silverman (1988) in four dimensions that depend on the way the information is processed: active vs. reflective, sensing vs. intuitive, visual vs. verbal and sequential vs. global.

They also mention Kolb's classification (1984) which is based on four dimensions confronted two by two: concrete experience vs. abstract conceptualization and active experimentation vs. reflective observation.

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[11] Oancea, R. et al (2018). Adaptivity in E-Learning Systems. In *International conference knowledge-based organization* (Vol. 24, No. 3, pp. 66-69). Sciendo.

1.1.2.2 Effective adaptive E-Learning systems according to learning style and knowledge level[12]

This article argues that adaptive learning systems should combine relevant characteristics of the learners, such as learning style and level of knowledge, to provide a more personalized learning experience.

In that sense, the article reports an investigation on how and when these factors should be used and evaluates their contribution to the improvement of learning based on the evaluation of the effectiveness of adaptive systems based on 174 subjects.

In turn, based on this assessment, the authors propose an adaptive approach built on a learning style model and a specific knowledge level; these are the same factors that will be used in this doctoral research to generate curricular adaptation.

1.1.2.3 Investigations about the effects and effectiveness of adaptivity for students with different learning styles[13]

In this study, with data from a sample of 147 students, the authors present the effects of adapting learning on different styles.

The article begins by stating that there is research that argues for the benefits of adaptive systems, and yet there are others that say that there are no differential effects.

The authors focused not on whether adaptive systems are effective, but on which learning style they might be most effective for. For this study, they divided students into three groups: one in

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[12] Alshammari, M. T., & Qtaish, A. (2019). Effective Adaptive E-Learning Systems According to Learning Style and Knowledge Level. *Journal of Information Technology Education*, *18, 529-547*.
[13] Graf, S. et al (2009). Investigations about the effects and effectiveness of adaptivity for students with different learning styles. In *2009 Ninth IEEE international conference on advanced learning technologies* (pp. 415-419). IEEE.

which learning styles agreed, another in which they did not, and a standard group in which all learning outcomes were offered regardless of learning style. The courses were geared towards learning object-oriented programming in an undergraduate program in information systems and computer science.

Based on the results, they concluded that positive effects are indeed noted in the learning of the students who were in the group where the learning styles agreed, over the other two groups, not so much on the learning outcomes, but on the ways of learning. They were also able to note that the adaptation of the learning system to different styles benefits active and reflective learners (according to the Felder and Silverman model) more than sequential and global learners.

In the article's recommendations, the authors propose that future work could analyze more deeply the interactions between the three dimensions of learning styles.

1.1.2.4 Adaptive education based on learning styles: are learning style instruments precise enough?[14]

The purpose of this research is to measure the impact of using different forms of information (visual and active) to construct instruments to characterize the learning style of students.

Based on this, the article concludes that the current instruments for measuring learning styles depend only on information presented in the form of text and that this generates a bias that is favorable to students who have verbal learning styles. To achieve this, the authors construct a new instrument for learning style measurement that uses different forms of information (figures, charts and equations) and, based on this instrument, make a first measurement to characterize the

[14] Alzain, A. et al (2018). Adaptive education based on learning styles: are learning style instruments precise enough? *International Journal of Emerging Technologies in Learning*, *13*(09), 41-52.

preferred learning style of 50 students. Then, they use the VARK instrument to make a second measurement and compare.

In the introduction to the article, the authors review research aimed at proving that a proper characterization of students' learning styles would not only benefit learning outcomes, but also motivation and engagement. These last two factors will be considered in this doctoral work as a way of measuring the impact of the redesigning of a curriculum based on adaptive learning.

The article makes an analytical comparison of different ways of classifying learning styles, which is very useful for deciding on which way should be used in this thesis. It also compares the way in which learning styles have been introduced in different systems of adaptive education.

Based on the results, the authors note that there is a high probability that the detection of students' learning styles is biased by the instruments with which it is measured.

1.1.2.5 Some ways of recognizing students' learning styles

This section describes, chronologically organized, the ideas presented in various articles aimed at showing ways of recognizing the learning styles of students.

First, Graf & Ives (2010)[15] present a flexible mechanism for providing adaptation based on learning styles for learning management systems. Learning management systems provide support for teachers in managing the content they make available to students. The authors propose introducing criteria based on learning styles to make these systems adaptive. To this purpose, they propose making the route of the courses more flexible, modifying the learning outcomes according to the

[15] Graf, S., & Ives, C. (2010). A flexible mechanism for providing adaptivity based on learning styles in learning management systems. In *2010 10th IEEE International Conference on Advanced Learning Technologies* (pp. 30-34). IEEE.

learning styles of the students. This is a basic but nevertheless valid way of providing adaptation to courses based on learning management systems.

For their part, Pham & Florea (2013)[16] take a step forward in determining students' learning styles. They propose determining these styles dynamically from student behavior in an online course on a learning management system. For this, they used the Felder and Silverman model, Index of Learning Style questionnaire, and showed how it improves accuracy in identifying students' learning styles. For this, they classified the learning objects into sixteen combinations of the four dimensions of learning styles, and also proposed variables associated with the time spent by the students in each of the proposed categories or combinations, and thus dynamically determined the learning style of each student.

Another form of dynamic determination of students' learning styles is proposed by Nafea (2018)[17]. This form dynamically reduces the number of questions in the Felder and Silverman learning styles questionnaire by restructuring the questionnaire into four groups arranged according to certain criteria, one for each learning style dimension. The student's learning style is determined from his or her response to the questions in the questionnaire, however, in the recommendations the author proposes the creation of an adaptive engine that builds the student's profile based on his or her learning behavior, knowledge, and performance.

Another step forward is taken by Li & Abdul Rahman (2018)[18] in proposing the detection of students' learning styles using tree augmented naive Bayes. These authors are not pioneers in using

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[16] Pham, Q. D., & Florea, A. M. (2013). A method for detection of learning styles in learning management systems. *UPB Scientific Bulletin, Series C: Electrical Engineering*, *75*(4), 3-12.

[17] Nafea, S. M. et al (2018, September). Ulearn: personalised learner's profile based on dynamic learning style questionnaire. In *Proceedings of SAI Intelligent Systems Conference* (pp. 1105-1124). Springer, Cham.

[18] Li, L. X., & Abdul Rahman, S. S. (2018). Students' learning style detection using tree augmented naive Bayes. *Royal Society open science*, 5(7), 172108.

the Bayesian approach to detect the learning style of students; however, they propose to use an augmented naive Bayesian network, which improves the accuracy of the classification. To use this method, they rely, as in the previously reviewed article, on the model of Felder and Silverman. In their article they describe the augmented naive method Bayesian network and compare it with the Bayesian network method, showing how the independence of the Bayesian network nodes from the first method causes their accuracy to be less than high and, therefore, they use the second method that maintains the characteristics of the previous method, but does not suppose this independence.

The article by Rasheed & Wahid (2019)[19] is a good example of a new trend in the identification of student learning styles, which is the application of neural networks for that purpose. In their research work, they use Gardner's theory of multiple intelligences, although they make some simplifications to make it more operational. Based on this model, they build and train a multiple layered neural network with back propagation that learns the learning style of each individual from the interactions he or she has with the network. As an additional contribution, in their article the authors make a concise but complete description of the various models for determining learning styles.

1.1.2.6 Some research showing evidence against the approach using learning styles

As well as having quite a few followers, learning styles and their detection models also have quite a few contradictors. Below are some articles that warn about the problems of using learning styles.

[19] Rasheed, F., & Wahid, A. (2019). Learning Style Recognition: A Neural Network Approach. In *First International Conference on Artificial Intelligence and Cognitive Computing* (pp. 301-312). Springer, Singapore.

Dembo & Howard (2007), in their article *Advice about the Use of Learning Styles: A Major Myth in Education* show how the use of learning styles has been challenged for many years, for example by Kampwirth & Bates (1980) and others more recently. The purpose of this article is not to do research to question the use of learning styles, but to open the dialogue of discussion between the two positions, in favor and against. The authors conclude that there is no consistent evidence that the use of teaching based on students' learning styles improves concentration, memory, self-confidence, performance or reduces anxiety. They say that educators continue to appeal to learning styles because it appears to promise a simple solution for solving educational problems. The authors advocate for other ideas related to improving learning, such as: teaching-learning strategies, systematically designed instruction that contains elements of scaffolding, or adaptation of instruction to different levels of prior knowledge.

For his part, Kirschner (2017) offers research results on the ineffectiveness of using models of learning styles and proposes that it would be more useful to rely on students' cognitive skills because they can be better producers of learning effectiveness. In his article, after reviewing several examples of research on the subject, he proposes the following four conclusions:

1. The premise that there are learners with different learning styles and that they should receive instruction using different instructional methods that match those styles is not a 'proven' fact, but rather a belief which is backed up by precious little, if any, scientific evidence.
2. There are many very fundamental problems regarding measuring learning styles.
3. The theoretical basis for the assumed interactions between learning styles and instructional methods is very thin.
4. Significant empirical evidence for the learning-styles hypothesis is almost non-existent.

These two articles, and all that they refer to, suggest that care should be taken with the indiscriminate use of learning styles as a factor in determining curricular adaptation.

1.1.3 Adaptive instructional systems

In this section articles that fall into in two categories will be reviewed: some on the state of the art regarding adaptive learning environments and others that propose examples of systems of this kind.

1.1.3.1 A Review on the Adaptive Features of E- Learning[20]

This article defines adaptive learning environments as those that can provide adaptive characteristics such as monitoring the activities of learners, interpreting their behavior based on specific domain models, inferring new requirements and preferences, and appropriately representing available knowledge in models that improve the learning process dynamically. It also proposes that contemporary strategies used in adaptive learning environments have been organized in four dimensions: curriculum sequencing, user modeling, adaptive navigation, and adaptive presentation, and then conducts a literature review around each of these dimensions.

The following figure shows a summary of the characteristics of each of the four dimensions.

[20] Datta, S., & Sengupta, S. (2018). A Review on the Adaptive Features of E-Learning. *International Journal of Learning and Teaching,* 4(4) 277-284.

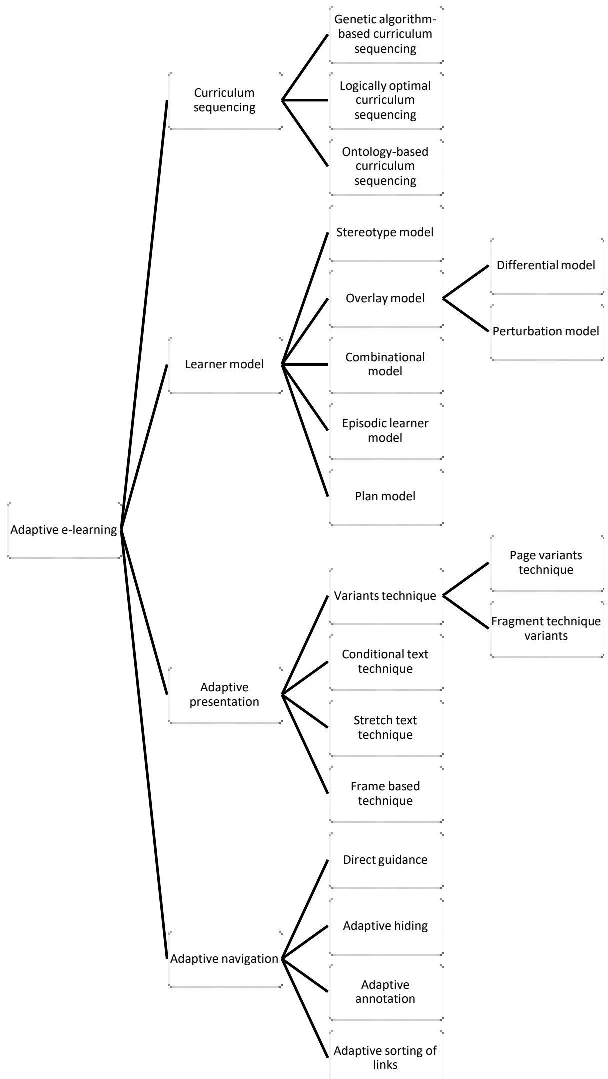

Figure 1. Hierarchical framework for adaptive features.

Datta, S., & Sengupta, S. (2018).[21]

The article describes each of these categories in detail. In the conclusions, the authors mention the rapid growth of this field and the constant emergence of new techniques in each of the four dimensions described.

[21] Datta, S., & Sengupta, S. (2018). A Review on the Adaptive Features of E-Learning. *International Journal of Learning and Teaching,* 4(4). p. 278.

1.1.3.2 Adaptive Learning Guidance System (ALGS)[22]

This poster presents the conceptual framework of the ALGS Adaptive Learning Guidance System.

The poster emphasizes that most adaptive learning systems marginalize the role of the teacher and proposes to equip the teacher with the ability to make contributions at all stages of the learning process.

This is based on a hybrid recommendation system with a three-stage architecture: collaborative filtering, to use data from other users in the classification of the student within the model; hybrid recommendation system, which uses data from the system and also from the teacher to adapt the student's route through the model; and the personalization engine, which uses the automatic data and also the data provided by the teacher to determine the routes and materials for learning.

1.1.3.3 A literature synthesis of personalised technology-enhanced learning: what works and why[23]

In this article the authors present a synthesis of literature published between 2000 and 2018 regarding the concept of technology-enhanced personalized learning (TEL). With this synthesis they intend to answer two questions: what are the potential benefits of TEL in secondary and higher education? and what are the contributions of TEL to the effectiveness of learning and teaching?

The authors begin by defining the concept of personalization referred to learning, that is, the ways in which learning can be used, based on certain characteristics to which apprentices give certain importance or relevance.

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[22] El-Hadad, G., Shawky, D., & Badawi, A. (2019). Adaptive Learning Guidance System (ALGS). *arXiv preprint arXiv:1911.06812*.

[23] FitzGerald, E. et al (2018). A literature synthesis of personalised technology-enhanced learning: what works and why. *Research in Learning Technology*, 26, article no. 2095.

They note that in 2004 the UK government's DfES (Department for Education and Skills) proposed five aspects to personalized learning:

1. Assessment for learning, to identify the needs of each learner.
2. Teaching and learning strategies, to develop skills and confidence in learners.
3. Curriculum entitlement and choice, which allows for choosing learning pathways from the system.
4. A student-centered approach, so that teachers can think creatively about how to support high quality learning and teaching.
5. Strong partnership beyond school, to remove barriers to learning.

On the potential benefits of personalized learning, they review information at three levels, that of the learner, that of the teacher and that of the institution. At the learner level, the most common benefits are increased motivation, empowerment and improved attitudes towards learning. They also emphasize that there is a potential for personalization to contribute to improved outcomes through improved student learning strategies. With respect to the potential benefits for teachers, they refer to some as the feedback that teachers can obtain from the adaptive system and the increase in the efficiency of the work, since it allows them to orient themselves to the highest-level comments. Finally, the benefits for the institution are mainly in the financial dimension, thanks to the use of educational resources in the long term, although some affirm exactly the opposite that educational institutions have an additional cost due to the implementation of adaptive systems.

Finally, in this systematic review, the authors present critiques and limitations of personalized learning in TEL. They refer to the fact that the identification of learning styles at the beginning of the process is not accurate enough to make a fair adaptation to the students and that, therefore, this process needs to be dynamic. They also show results from studies that assert that personalization is not always necessary for productive learning environments and that, as noted above, it can be costly in financial terms and in terms of time and resources. They also relate the paradox of choice-personalization, which refers to the fact that the more resources are available, the more difficult it is to look for those that are of good quality to personalize for student learning.

This article, rather than specifically describing the characteristics of personalized learning environments, is devoted to looking at aspects related to their implementation.

1.1.3.4 Selected examples of adaptive instructional systems

Below is a table summarizing the main characteristics of some learning environments selected for their relevance. For ease of representation, a code will be assigned to each of the articles reviewing these environments, as follows:

1. An adaptive educational system for higher education[24]
2. A multimedia adaptive tutoring system for mathematics that addresses cognition, metacognition and affect[25]
3. Una plataforma para la implementación de cursos en línea adaptativos: descripción y punto de vista de los docentes[26]
4. Use of Felder and Silverman learning style model for online course design[27]

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[24] Martins, C. et al (2008). An adaptive educational system for higher education. *Proceedings of the 14th EUNIS*, *8*, 24-27.
[25] Arroyo, I. et al (2014). A multimedia adaptive tutoring system for mathematics that addresses cognition, metacognition and affect. *International Journal of Artificial Intelligence in Education*, *24*(4), 387-426.
[26] Moreno, J. et al (2014). Una plataforma para la implementación de cursos en línea adaptativos: descripción y punto de vista de los docentes. *Revista Electrónica de Investigación Educativa*, 16(3), 103-117.
[27] El-Bishouty, M. M. et al (2019). Use of Felder and Silverman learning style model for online course design. *Educational Technology Research and Development*, *67*(1), 161-177.

5. An empirical study on the impact of using an adaptive e-learning environment based on learner's personality and emotion[28]
6. Adaptive learning system and its promise for improving student learning[29]
7. Learning in smart environments: user-centered design and analytics of an adaptive learning system[30]
8. KLSAS – an adaptive dynamic learning environment based on knowledge level and learning style[31]
9. An Adaptive Recommender-System Based Framework for Personalised Teaching and Learning on E-Learning Platforms[32]

Table 1. Summary of features of adaptive instructional systems.

Author's own elaboration.

Feature	1	2	3	4	5	6	7	8	9
Level of schooling[33]	H	K	NA	H	H	K	H	H	NA
Made on Moodle or other LMS	Y	N	N	Y	N	N	N	N	N
Development by programming	N	Y	Y	N	Y	Y	Y	Y	Y
Adaptation criteria[34]	KBIL	KEA	K	L	PM	K	K	KL	KL
Dynamic adaptation	N	Y	N	N	N	Y	Y	N	Y
Adaptation system[35]	NB	B	NB	NB	NB	NB	NB	NB	B
Course taught[36]	P	M	NA	E	P	LS	P	P	NA

The Wayang Outpost system outlined in Arroyo, I. et al (2014) deserves a special note, as it has a configuration that requires students to identify the type of mathematical problem before beginning

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[28] Fatahi, S., & Moradian, S. (2018). An Empirical Study on the Impact of Using an Adaptive e-Learning Environment Based on Learner's Personality and Emotion. *International Association for Development of the Information Society*.

[29] Li, H. et al (2018) Adaptive Learning System and Its Promise on Improving Student Learning. In *Proceedings of the 10th International Conference on Computer Supported Education (CSEDU 2018) 2*, 45-52

[30] Vesin, B. et al (2018). Learning in smart environments: user-centered design and analytics of an adaptive learning system. *Smart Learning Environments*, *5*(1), 24.

[31] Dhakshinamoorthy, A., & Dhakshinamoorthy, K. (2019). KLSAS—An adaptive dynamic learning environment based on knowledge level and learning style. *Computer Applications in Engineering Education*, *27*(2), 319-331.

[32] Maravanyika, M. et al (2017). An adaptive recommender-system based framework for personalised teaching and learning on e-learning platforms. In *2017 IST-Africa Week Conference (IST-Africa)* (pp. 1-9). IEEE.

[33] H: Higher Education; S: K-12.

[34] K: Knowledge/Cognition; B: Behavior; I: Interests; L: Learning Styles; E: Engagement; A: Affect; P: Personality: M: Emotion.

[35] B: Bayesian; NB: No Bayesian; NA: Not applicable.

[36] P: Computer Programming; M: Mathematics; E: Related to education; L: Language; S: Sciences.

to solve it. This can be a very useful feature for the adaptive system that will be used in this thesis; in this case, this identification could be made based on the ways of thinking of the mathematical areas to which the adaptation is particularly directed at each moment. Also interesting is the coaching system that this adaptive learning environment has, which bases its support on several characteristics, among them and in a special way, on metacognition.

1.2 Curriculum

Another relevant topic for this doctoral research is related to the curriculum. This section reviews articles that describe current trends in curriculum design, ergonomics, and adaptation parameters.

1.2.1 Curriculum design

This thesis aims at developing a framework for curriculum design. The following articles show the status of curriculum design.

1.2.1.1 Task analysis in curriculum design: a hierarchically sequenced introductory mathematics curriculum[37]

Although this article dates from 1973, in it the authors present an approach that is still valid in curricular design: task analysis, which is also useful for tasks of adaptation to the various characteristics of students.

This article is an intermediate result of a research program that explores in detail the procedures for task analysis. The purpose of the article is to develop a systematic method for specifying and validating hierarchies in which instructional programs can be designed to provide an optimal match

[37] Resnick, L. B. et al (1973). Task analysis in curriculum design: a hierarchically sequenced introductory mathematics curriculum. *Journal of Applied Behavior Analysis*, *6*(4), 679-709.

for the students' natural learning sequence. This purpose coincides very closely with the logic of learning adaptation, although when it was formulated, this concept was not yet developed.

The method proposed by the authors consists in developing hierarchies of more specific learning objectives (simpler tasks) that facilitate the learning of higher objectives (more complex tasks). This method implies the identification of concepts included in the curriculum and the establishment of a hierarchy of prerequisites among them, to allow the student to reach the acquisition of competence.

The sequencing of tasks is not only given by cognitive prerequisites, but also by differences in other basic functions such as memory or perceptual organization, so that these tasks can be organized according to their complexity.

Another interesting contribution of the article is a form of graphic convention which allows the sequencing of tasks to be recognized.

Task analysis is an important tool for the process of curricular adaptation proposed in this doctoral research.

1.2.1.2 El método de análisis teórico de la actividad: una alternativa para el diseño curricular[38]

This article presents two implementations of the method of theoretical analysis of activity for curriculum design.

A definition of the concept of curriculum, provided by González (1995), is presented: *"A training project and a process of realization through a structured and ordered series of contents and learning experiences,*

[38] Hernández Díaz, A. et al (2018). El método de análisis teórico de la actividad: una alternativa para el diseño curricular. *Dilemas Contemporáneos: Educación, Política y Valores*, *6*(1).

articulated in the form of a political-educational proposal that is proposed by diverse social sectors interested in a particular type of education, with the purpose of producing learning that is translated into ways of thinking, feeling, and acting in the face of complex problems posed by social life in a given country".[39] This definition highlights the fact that the curriculum implies sequencing learning contents and experiences in a certain order, and although it is not expressed in adaptive terms, it allows the adaptive to fit into it. The authors put it this way: *"This conception is distinguished by its character of project and process, which gives it a high dose of flexibility and the possibility of adjustment to the context in which it is developed and induces us to contemplate three independent, but at the same time very interrelated, moments within curricular work:*

- *Curricular design (phase of construction and initial theoretical reflection, where the results to be achieved are foreseen),*
- *implementation or development (the carrying out of each of the actions foreseen in the design, under the specific conditions of the educational context) and*
- *monitoring and evaluation (accompaniment during the process at both orientation and implementation) to identify possible adjustments if required at the end of the process*".[40]

The method of theoretical analysis of the activity for curriculum design, a product of educational research in Cuba over the last two decades, proposes that the curriculum should be designed based on the needs of the students' incoming profile and the possibilities with which they enter the programs. In this sense, this method is useful for this doctoral research since it establishes a practical methodology for the detection of these two elements, which are key to the adaptive process.

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39 Hernández Díaz, A. et al (2018). El método de análisis teórico de la actividad: una alternativa para el diseño curricular. *Dilemas Contemporáneos: Educación, Política y Valores*, *6*(1). p. 10.
40 Hernández Díaz, A. et al (2018). El método de análisis teórico de la actividad: una alternativa para el diseño curricular. *Dilemas Contemporáneos: Educación, Política y Valores*, *6*(1). p. 4.

1.2.1.3 Collaborative Curriculum Design for Sustainable Innovation and Teacher Learning[41]

This book presents the results of various investigations framed in four paradigms of curriculum design: the instrumental paradigm, the communicative paradigm, the artistic paradigm, and the pragmatic paradigm.

These four paradigms are substantiated and then exemplified with various cases throughout the book.

In general, the book seeks to show the importance of actively including teachers in the design of the curriculum, not only because of what these actors can contribute to the quality of the curriculum, but also because of the feedback that such participation gives to teacher training itself.

The book also shows that there are two different kinds of curriculum materials, depending on how teachers interact with them. The first kind is objective, in which materials are used according to their own design characteristics; and the other kind is subjective, in which materials are used depending on teaching needs and not necessarily according to the original design characteristics.

The book argues for the creation of teams of teachers to design curricula that are more relevant to the needs of the systems. It also demonstrates that through the active participation of teachers in curriculum design and the development of curriculum materials, teachers take responsibility for how these materials transform their teaching and can contribute to student learning.

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[41] Pieters, J., Voogt, J., & Pareja Roblin, N. (2019). *Collaborative curriculum design for sustainable innovation and teacher learning* (p. 424). Springer Nature. (Pieters & Pareja Roblin, 2019)

1.2.2 Adaptive curriculum

Studies pertaining to adaptive curricula is an active current field of research. The following is a review of research on this topic.

1.2.2.1 Uma Abordagem Evolutiva para o Problema de Sequenciamento Curricular Adaptativo[42]

As the article says, Adaptive Curriculum Sequencing (ACS) is still a challenge in the adaptive learning field. To contribute to the field, this article proposes working in the context of the prey-predator algorithm.

The authors explain that the objective curricular sequencing can be seen as a function $f(a, d, e) \rightarrow S$, which receives as parameters the model of the student (*a*), the information of the didactic material (*d*) and the information of the pedagogical structure of the concepts (*e*) and returns a sequence that best approximates the requirements of the three parameters.

Although the purpose of this doctoral thesis does not go as far as the development of a technical solution for the automatic sequencing of the curriculum but is more related to pedagogical characteristics and mathematical content, the work done in this research and reported in this article is nevertheless interesting.

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[42] Machado, M. et al (2018). Uma Abordagem Evolutiva para o Problema de Sequenciamento Curricular Adaptativo. In *Brazilian Symposium on Computers in Education (Simpósio Brasileiro de Informática na Educação-SBIE)* (Vol. 29, No. 1, p. 1243).

1.2.2.2 Integrating Human and Machine Intelligence for Enhanced Curriculum Design[43]

This document is a doctoral thesis proposing semi-automatic techniques for curriculum design that combine machine learning, human computing, and principles of the science of learning.

The fundamental contribution of the thesis is the attention to the problem of content sequencing for students. This is done using multiple learning models and the combination of these models with psychological theory.

The author of the thesis combines several fields and articulates them with instructional design and computational systems, thus achieving a curriculum created by the interaction between machine intelligence and human intelligence.

1.3 Mathematical thinking

One of the most important concepts dealt with in this doctoral thesis is mathematical thinking. Some relevant texts in this regard are reviewed below.

1.3.1 Competencies and mathematical learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark[44]

This book contains a report of the Competence Development and Mathematics Learning project which sought to answer the following questions in the Danish context:

> *"a) To what extent is there a need for a renewal of existing forms of mathematics teaching?*

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[43] Doroudi, S. (2019). *Integrating Human and Machine Intelligence for Enhanced Curriculum Design* (Doctoral dissertation, Air Force Research Laboratory).

[44] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485.

b) Which mathematical competencies need to be developed in students at the different stages of the education system?

c) How does one ensure progression and coherence in mathematics teaching throughout the education system?

d) How does one measure mathematical competencies?

e) What should be the content of up-to-date mathematics teaching?

f) How does one ensure the ongoing development of mathematics as a subject and of mathematics teaching?

g) What does society demand of mathematics teaching?

h) What do future mathematical teaching materials look like?

i) How can one, in Denmark, make use of international experience with mathematics teaching?

j) How should mathematics teaching be organised in the future?" [45]

In summary, what the KOM (Knowledge of Mathematics) project aims to achieve is *"to produce an adequate characterisation of mathematical subject specialisation based on mathematical competences as a means of meeting some of the challenges and dealing with some of the problems"*.[46]

In the development of the project the working group proposes a competence description of mathematical education in these terms: *"mathematical competency is a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge"*.[47]

[45] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485. p. 15.
[46] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485. p. 20.
[47] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485. p. 49.

This definition of mathematical competence led the group to propose eight core competencies in mathematics. These competencies can be seen in the diagram.

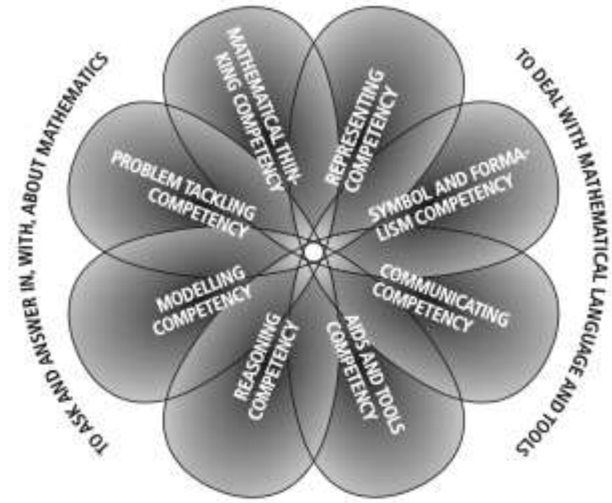

Figure 2. A visual representation of the eight mathematical competencies.

Niss, M. A., & Højgaard, T. (Eds.) (2011).[48]

The eight competencies are divided in two categories. One, which includes the first four competencies, refers to the ability to formulate and answer questions in and with mathematics; and the other, which includes the other four competencies, refers to the ability to handle language and mathematical tools.

On the other hand, the report deals with the training of mathematics teachers, in terms of the definition and classification of competencies described above.

The report also refers to the interplay between subject matter and the competencies. In this regard, the authors propose that *"a competency can be practised in relation to the given subject material, i.e., come into play and be expressed in relation to this subject material. A competency can be developed, i.e., created or consolidated*

[48] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485. p. 51.

in relation to the given subject material".[49] These two types of interaction lead to the creation of a matrix that allows the selection of adequate resources for the development and practice of competencies. This matrix can be seen as follows.

Competency/ Subject area	Math. thinking comp.	Problem tackling comp.	Model-ling comp.	...	Aids and tools comp.
subject area 1					
subject area 2					
...					
subject area n					

Figure 3. Relationship matrix between themes, competencies, and resources.
Niss, M. A., & Højgaard, T. (Eds.) (2011).[50]

The text describes the methodology for selecting the topics in each area in relation to the competencies and the relevant materials for developing and teaching the topics. This way of proceeding can be very useful for this doctoral thesis.

Finally, the report proposes methodologies for competence assessment. In this part, the authors state that competences are manifested in activities and that, therefore, evaluation should be done through these. They also specifically define what mathematical activity means: *"a set of conscious and goal oriented mathematical actions in a situation"*.[51] Based on the activities, one can determine whether a person has the mastery of a certain competency and one can also determine the competency profile of a certain person. The latter gives rise in the report to the characterization of the progression in

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[49] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485. p. 123.
[50] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485. p. 125.
[51] Niss, M. A., & Højgaard, T. (Eds.) (2011). Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark. Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, No. 485. p. 137.

mathematical competency, which is an important concept for adaptive learning, the focus of the present thesis.

Regarding evaluation, it proposes new forms of evaluation and advocates continuous evaluation.

1.3.2 How humans learn to think mathematically: Exploring the three worlds of mathematics[52]

In this book Tall explains the theory of the three worlds of mathematics, a framework for the cognitive development of mathematical thinking, where each world is based on human perception, action, and reason.

The three worlds of mathematics are described as follows.

- The embodied world explores the perceptual properties of physical and mental objects, formulating verbal definitions used at a more sophisticated level to reason about relationships.
- The symbolic world develops out of mathematical operations performed initially on real world objects, where the operations are symbolized and the symbols themselves are manipulated as mental objects. These symbols can then be operated upon at successively higher levels in arithmetic, algebra, symbolic calculus, vector algebra and so on.
- The formal world is based on verbal/logical definition with properties deduced by mathematical proof.

In each world there is a long-term development in sophistication:

[52] Tall, D. (2013). How humans learn to think mathematically: Exploring the three worlds of mathematics. Cambridge University Press.

- Objects develop sophisticated structure.
- Operations are symbolized and the symbols may then be conceived as mental objects with structure.
- Properties are later formulated verbally to define formal concepts whose other properties are deduced by formal proof.

The book describes how these three worlds can be seen and the developments on which they depend, both in school mathematics and in university mathematics.

1.3.3 Understanding the Mathematical Way of Thinking – The Registers of Semiotic Representations[53]

In this book, which is the product of a seminar for doctoral students in the mathematics education program at the Universidade Bandeirante de São Paulo, Duval explains his theory of semiotic registers, which is a construct for analyzing cognition in students who are developing their mathematical thinking. Fundamentally, this theory proposes that mathematical objects have various forms of representation that must be ordered to ensure students' understanding of these objects.

The central idea that this book develops is: *"Semiosis is at the center of the cognitive processes of mathematical thinking through two kinds of transformations of semiotic representation. There is no noesis without semiosis, no mathematical thinking without transformation of semiotic representations whatever they are"*.[54]

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[53] Duval, R. (2017). Understanding the Mathematical Way of Thinking–The Registers of Semiotic Representations. Springer International Publishing.
[54] Duval, R. (2017). Understanding the Mathematical Way of Thinking–The Registers of Semiotic Representations. Springer International Publishing. p. 22.

The author, perhaps because the book is the product of a seminar for the teaching of this subject to teachers, fills it with examples based on different mathematical objects, from the most obvious in their representation to others in which changes of representation are less usual.

An interesting chapter for this doctoral thesis is the one dealing with semiotic registers, as a method of identification and analysis of cognitive variables. This method contains relevant elements for analyzing students' learning pathways, which can lead to adaptive behaviors that can be modeled in a system.

1.3.4 Assessing mathematical thinking as part of curriculum reform in the Netherlands[55]

This article reports research showing the impact of curriculum reform on the assessment of mathematical thinking. The curricular reform in question was implemented in the Netherlands between 2011 and 2017, in some pilot schools, while in others the established curriculum was maintained.

The study aimed at answering the following question: *"How is the curriculum reform with respect to mathematical thinking reflected in national examinations papers in the Netherlands and in student performance on corresponding assignments?"* [56] And the results showed that schools that implemented the new curriculum assessed mathematical thinking to a greater degree than schools with the regular curriculum; however, this impetus for assessment decreased over time. The authors formulate the hypothesis that conservative forces made the traditional curriculum push back from what was established by the new curriculum.

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[55] Drijvers, P. et al (2019). Assessing mathematical thinking as part of curriculum reform in the Netherlands. *Educational Studies in Mathematics*, 1-22.
[56] Drijvers, P. et al (2019). Assessing mathematical thinking as part of curriculum reform in the Netherlands. *Educational Studies in Mathematics*, p. 13.

The theoretical framework of the article describes the four notions about assessment as part of the implementation of curriculum reform that the authors took into account in their analysis: the validity of assessment, the fact that assessment should reflect teaching and learning as they took place in reform education, a taxonomy to classify learning goals, and, finally, Kuiper's model of incentives to implement curriculum reform in order to position assessment in the implementation of curriculum reform as a whole.

Also, within this theoretical framework, the authors describe mathematical thinking and its purpose. In this sense, they cite Pólya (1963) who says: *"First and foremost, it should teach those young people to think"*.[57] They describe the elements of mathematical thinking that they analyzed in the reform of the curriculum: problem solving, modeling, and abstraction; the last two of these elements were analyzed both seen as processes and seen as objects. The model of analysis they used can be seen in the figure. This model is an adaptation of the model proposed by Drijvers (2015).

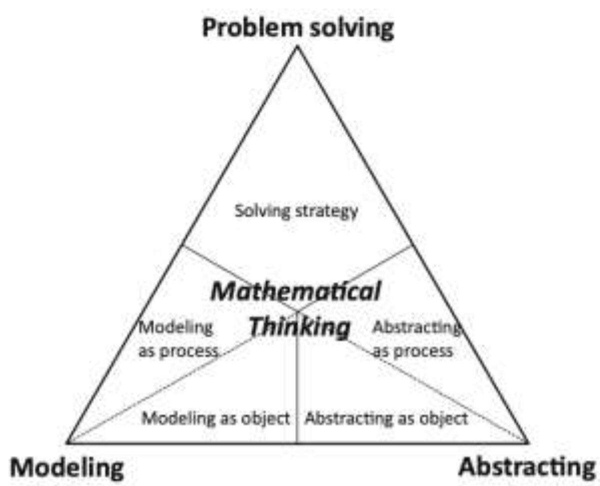

Figure 4. Model used for analysis of evaluation documents.

Drijvers, P. et al (2019).[58]

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[57] Polya, G. (1963). On learning, teaching, and learning teaching. *The American Mathematical Monthly*, *70*(6), 605-619. p. 605.

[58] Drijvers, P. et al (2019). Assessing mathematical thinking as part of curriculum reform in the Netherlands. *Educational Studies in Mathematics*, p. 8.

1.4 Teaching linear algebra

The practical contribution of this doctoral thesis involves the implementation in the teaching of linear algebra of an adaptive curriculum focused on the development of mathematical thinking. Therefore, it is important to review the state of the art in the teaching of this branch of mathematics.

1.4.1 On some aspects of students' thinking in linear algebra[59]

In this article, Sierpinska presents, among other issues, three modes of reasoning in linear algebra: synthetic-geometric, analytic-arithmetic, and analytic-structural.

This description of the modes of reasoning is based on experiments carried out between 1996 and 1999 in which, together with other researchers, she proposes to overcome the 'obstacle of formalism'. In the article, she reports that despite the efforts put into the experiments the difficulties persisted, which led the researchers to think that the problem was in the fact that they continued to present the same theory, that is, the structural theory of linear algebra with the axiomatic definition of linear transformations. The students could not understand this theory because they had their minds more focused on practical thinking than on theoretical thinking. From this understanding, she proposed three modes of reasoning.

In the first part of the chapter, the author presents the difficulties produced by practical thinking in learning linear algebra. Difficulties such as: the transparency of language, the difficulty of understanding definitions, the approach to mathematical concepts in terms of prototypical examples instead of definitions, reasoning based on the logic of action and generalization from visual perception, are described and illustrated.

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[59] Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209-246). Springer, Dordrecht.

Then, in the second part of the chapter, the three modes of thinking in linear algebra are described. Sierpinska states that *"while these modes of thinking appeared in the history of mathematics in a sequential manner, it did not happen that one of them eliminated the other two"*.[60]

She explains why the modes are called synthetic and analytical. *"In the synthetic mode the objects are, in a sense, given directly to the mind which then tries to describe them, while in the analytic mode they are given indirectly, in fact, they are only constructed by the definition of the properties of their elements"*. [61]

The synthetic-geometric mode of thinking uses the language of geometric figures - planes and lines, intersections - as well as their conventional graphical representations. *"In the analytic-arithmetic mode, geometric figures are understood as sets of 'n-tuples' of numbers satisfying certain conditions that are written"*.[62] While *"analytic-structural thinking goes beyond this type of analysis and synthesizes the algebraic elements of the analytic representations into structural wholes"*.[63]

This article is of crucial importance for the development of the research reported in this doctoral thesis.

1.4.2 ¿Cómo se aprenden los conceptos de álgebra lineal?[64]

This article presents the results of an investigation, part of a program that researches, from the point of view of APOS theory, the constructions involved in the different concepts of linear algebra,

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[60] Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209-246). Springer, Dordrecht. p. 232.
[61] Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209-246). Springer, Dordrecht. p. 233.
[62] Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209-246). Springer, Dordrecht. p. 234.
[63] Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209-246). Springer, Dordrecht. p. 235.
[64] Oktaç, A., & Trigueros, M. (2010). ¿Cómo se aprenden los conceptos de álgebra lineal? *Revista Latinoamericana de Investigación en Matemática Educativa, RELIME*, *13*(4), 373-385.

in particular the concepts of vector space, linear transformation, base and systems of linear equations.

The authors use APOS theory to effect a process of genetic decomposition of the concepts, which later allows the design and application of teaching strategies. They also review work done by RUMEC (Research in Undergraduate Mathematics Education Community), a group dedicated to doing research using APOS theory, for an introductory linear algebra course (Weller et al., 2002).

The process of genetic decomposition proves potentially useful for the development of the domain model of an adaptive system such as the one to be used in this doctoral thesis.

1.4.3 Student connections of linear algebra concepts: an analysis of concept maps[65]

In this article, researchers explore the connections between linear algebra concepts developed in a first undergraduate level course, based on the analysis of concept maps made by students relating the concepts they learned during the course.

For this exploration, the authors rely on quantitative and qualitative analysis of concept maps; however, they emphasize that although quantitative measures give some insight into students' understanding of concepts, they have limitations. Maps that have the same score can give different ideas about students' understanding. In terms of qualitative measures, the article reports the use of clumps as a strategy for determining strength in the relationship between different concepts. Non-directed graphs and adjacency matrices are also used for analysis.

In their analysis, they find five groups of concepts or five clumps: the manipulation of matrices and systems of equations, bases and dimensions, similarity and transformations, Eigenvalues and

[65] Lapp, D. A., et al (2010). Student connections of linear algebra concepts: an analysis of concept maps. *International Journal of Mathematical Education in Science and Technology*, *41*(1), 1-18.

Eigenvectors, and orthogonality. Based on this, they conclude that these clumps are due to the common way of teaching and shaping the curriculum. However, they also note that the way in which students link these clumps together is not the same, which sheds light on recognizing their forms of understanding. As a result of this analysis, the researchers emphasize that the concepts of Eigenvalues and Eigenvectors are the least related to the rest of the concepts of linear algebra.

The kind of analysis carried out in this research can be useful in developing the domain model of an adaptive system if it is done from the conceptions of teachers.

1.4.4 Un cambio metodológico y de contenidos en álgebra lineal[66]

The article presents the results of research carried out at the Department of Engineering and Technological Research of the Universidad Nacional de La Matanza in Argentina in relation to the methodological and content changes in the subject Algebra and Analytical Geometry I.

This reform of the programmatic contents was generated from an integrative and relational point of view, in agreement with what was proposed by the Linear Algebra Curriculum Study Group: *"a) The contents and their presentation must respond to the needs of the client disciplines (Physics, Engineering, Economics, etc.). Generalization and deepening should be done as time allows. b) A first LA course should be strongly oriented to matrices. Less abstraction is suggested (together with), more emphasis on problem solving and motivating applications, going from the concrete to the more conceptual. c) Consider the needs and interests of students as learners. Seek an active teaching-learning methodology, without forgetting the students' previous knowledge. d) Use technology. e) Develop a second LA course with a more conceptual bias and greater theoretical justification"*.[67]

[66] Bertúa, J., & Denenberg, M. (2016) Un cambio metodológico y de contenidos en álgebra lineal. *Revista de Educación Andrés Bello*, No. 4, pp.54-86.
[67] Bertúa, J., & Denenberg, M. (2016) Un cambio metodológico y de contenidos en álgebra lineal. *Revista de Educación Andrés Bello*, No. 4, p. 60.

Given the justification of the research project reported in the present doctoral thesis, it is considered that this may be an adequate approach framing the curriculum to be implemented as a practical contribution, since an adaptive approach is characteristic of subjects designed for a wide diversity of client disciplines (using the terminology of the article).

The article does not report measured results of the impact of the reform in terms of student performance, however, it does report conclusions on the methodology of implementation.

1.4.5 Challenges and Strategies in Teaching Linear Algebra[68]

This book is a monograph of ICME 13, held in Hamburg in 2016. It presents the state of the art of research related to the teaching of linear algebra. It consists of four parts: theoretical perspectives elaborated through tasks, analyses of learners' approaches and resources, dynamic geometry approaches, and challenging tasks with pedagogy in mind. For the purposes of this doctoral thesis, the chapters in the first part and some chapters in the fourth part are relevant.

In the first part, there are three chapters:

1. The learning and teaching of linear algebra through the lenses of intellectual need and epistemological justification and their constituents[69]. Guershon Harel.
2. Learning linear algebra using models and conceptual activities[70]. María Trigueros.

[68] Stewart, S. et al (Eds.). (2018). *Challenges and strategies in teaching linear algebra*. Springer. Cham.
[69] Harel, G. (2018). The Learning and Teaching of Linear Algebra Through the Lenses of Intellectual Need and Epistemological Justification and Their Constituents. In *Challenges and Strategies in Teaching Linear Algebra* (pp. 3-27). Springer, Cham.
[70] Trigueros, M. (2018). Learning linear algebra using models and conceptual activities. In *Challenges and strategies in teaching linear algebra* (pp. 29-50). Springer, Cham.

3. Moving between the embodied, symbolic, and formal worlds of mathematical thinking with specific linear algebra tasks[71]. Sepideh Stewart.

In the first, Harel analyzes the implications of intellectual needs and epistemological justification, two central constructs in his conceptual framework called DNR-based instruction in mathematics, for curriculum development and linear algebra teaching. The author presents two classification systems for the intellectual needs of students. The first has two categories: local needs and global needs; and the second has five categories: need for certainty, need for causality, need for computation, need for communication and formalization, and need for structure. Likewise, epistemological justification is classified into three categories: sentential epistemological justification (SEJ), apodictic epistemological justification (ASJ), and meta epistemological justification (MEJ).

Before presenting these classifications, Harel summarizes the definitions of the concepts central to his theory of mathematical learning: ways of understanding and ways of thinking.

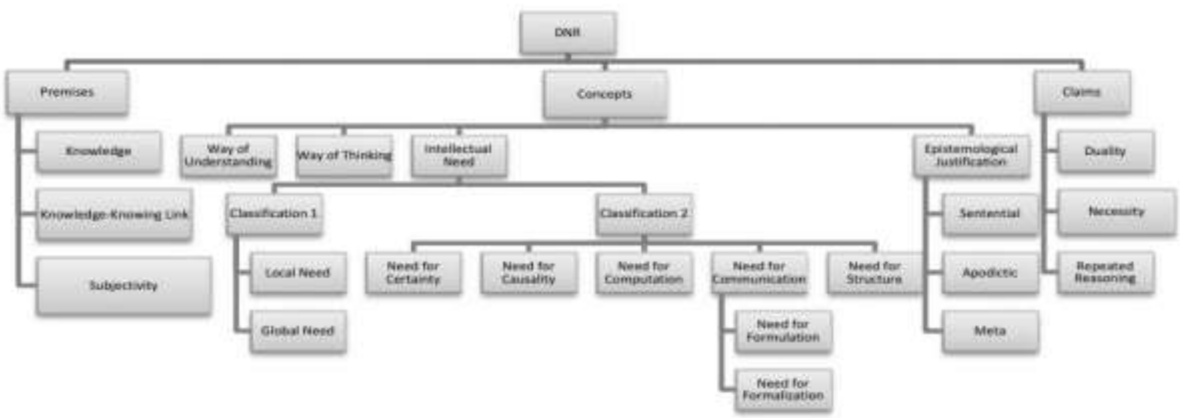

Figure 5. Classification of epistemological justification and the third foundational principle of DNR.

Harel, G. (2018).[72]

[71]Stewart, S. (2018). Moving between the embodied, symbolic and formal worlds of mathematical thinking with specific linear algebra tasks. In *Challenges and strategies in teaching linear algebra* (pp. 51-67). Springer, Cham.
[72] Harel, G. (2018). The Learning and Teaching of Linear Algebra Through the Lenses of Intellectual Need and Epistemological Justification and Their Constituents. In *Challenges and Strategies in Teaching Linear Algebra* (pp. 3-27). Springer, Cham. p. 16.

The chapter provides examples of these theoretical elements in the learning and teaching of linear algebra.

For her part, Trigueros relies on APOS theory to propose an innovative approach that includes challenging modeling situations. She also shows successful cases of application of this approach, focused on three important concepts of linear algebra: systems of linear equations, linear independence and Eigenvalues, Eigenvectors and Eigenspaces. These success stories are reported as moments in which students change their ways of thinking (their schemes). The research reported in this chapter shows common phenomena in learning that appear independently of the concepts and activities that were designed for teaching.

Finally, in the third chapter, Stewart, based on the results of some of the research he has carried out, presents specific tasks to enable students to move between the three worlds proposed by Tall's model (embodied, symbolic, and formal).

The author states in his article that for more than a decade he has used Tall's model along with Dubinsky's APOS model to build a framework called Framework of Advanced Mathematical Thinking (FAMT) to investigate students' conceptual understanding of linear algebra concepts. He says that *"The natural blend of these two learning theories provides an ideal platform to analyse students' thinking in the context of primary concepts in linear algebra (e.g., vectors, linear combinations, linear independence, basis, range and eigenvalues and eigen-values)".*[73]

In addition, he highlights the importance of visualization in mathematical education and shows a compilation of studies related to this topic.

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[73] Stewart, S. et al (Eds.). (2018). *Challenges and strategies in teaching linear algebra*. Springer. Cham. p. 51.

With respect to the articles in part four, the chapters highlighted are:

1. Linear Algebra - A companion of advancement in mathematical comprehension[74].
2. Using Challenging Problems in Teaching Linear Algebra[75].

Kobal proposes that linear algebra would be successfully taught if it is presented as a tool to master various mathematical problems. Using a problem-based approach in his article, the author presents examples of challenging problems that use linear algebra.

Kobal contradicts the position that linear algebra is a branch of mathematics that explores vector spaces and linear maps between them and, rather, proposes that these advanced concepts of the subject be developed gradually, beginning by showing the student that linear algebra is the study of line-like relationships, which gradually evolve into more complex concepts.

The chapter is intended for teachers of linear algebra to help them teach the subject in a challenging way that motivates students rather than frustrating them. To solve problems of a geometric nature, the author makes use of concepts of linear algebra.

It is important to emphasize the idea that the author expresses as the conclusion of the chapter: *"Teaching mathematics has always been about good presentation of good ideas. Linear algebra offers many wonderful and smart mathematical ideas, which combine visual and analytical thinking, which offer a smooth transition from concreteness to abstraction, and which appear on crossroads of all mathematical fields. This is a great challenge and a valuable opportunity, which a devoted teacher should not miss".*[76]

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[74] Kobal, D. (2018). Linear Algebra—A Companion of Advancement in Mathematical Comprehension. In *Challenges and Strategies in Teaching Linear Algebra* (pp. 279-298). Springer, Cham.
[75] Berman, A. (2018). Using Challenging Problems in Teaching Linear Algebra. In *Challenges and Strategies in Teaching Linear Algebra* (pp. 369-378). Springer, Cham.
[76] Kobal, D. (2018). Linear Algebra—A Companion of Advancement in Mathematical Comprehension. In *Challenges and Strategies in Teaching Linear Algebra* (pp. 279-298). Springer, Cham. p. 297.

In his chapter, Berman presents examples of challenging problems that he says can make the teaching and learning of linear algebra enjoyable. Problems such as these can be useful for the design of the resources of the adaptive model which is the object of this doctoral thesis.

1.4.6 Linear algebra learning focused on plausible reasoning in engineering programs[77]

In this article the authors present a teaching strategy for linear algebra in engineering programs based on plausible reasoning. In addition to this concept, this strategy uses other concepts such as: non-routine problems, geometric visualization, and conjecture generation, as the basis for the formulation of the project.

This strategy was formulated by the authors as a solution to the research question: *"How to affect different factors that influence negatively the betterment of linear algebra teaching and learning process?"*.[78]

The proposed strategy has four parts: the theoretical foundations of linear algebra, the diagnosis that allowed the generation of a model of solving problems through plausible reasoning implemented in software (Geogebra), the factors that accelerate the solution of the problem, and the implementation.

1.4.7 Conceptual understanding of dot product of vectors in a dynamic geometry environment[79]

This article points out that the most recent trends in mathematics education related to the use of technology emphasize students' conceptual understanding rather than procedural understanding,

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[77] García-Hurtado, O. et al (2019). Linear algebra learning focused on plausible reasoning in engineering programs. *Visión electrónica*, *13*(2).
[78] García-Hurtado, O. et al (2019). Linear algebra learning focused on plausible reasoning in engineering programs. *Visión electrónica*, *13*(2). p. 4.
[79] Donevska-Todorova, A. (2015). Conceptual Understanding of Dot Product of Vectors in a Dynamic Geometry Environment. *Electronic Journal of Mathematics & Technology*, *9*(3).

which was emphasized earlier. And although the article does not use adaptive technology, but rather dynamic geometry, it also relies on modes of thinking to achieve changes in representation, thus generating development of students' mathematical thinking.

After presenting a definition of conceptual understanding and specific characteristics by which it can be recognized that this type of understanding exists in students, specifically in linear algebra (distinguishing what is and what is not the concept, concept definitions and concept images, multiple modes of description, language and thinking, concept's properties which construct an axiomatic definition of the concept, connections of the concept with other concepts), the author proposes ideas related to the role that technology has in the development of this type of understanding, specifically in linear algebra. She puts this proposal concretely in the work of teaching a group of students the concept of the dot product.

In addition to the general conception of conceptual understanding, this article is useful for this research in terms of the way it develops the concept of the dot product.

Chapter conclusions

This chapter contains the results of the exploration of the state of the art in the fields related to the present doctoral research work. Based on this exploration it can be concluded that:

- The field of adaptive learning is a very active area of study today. In general, it seeks to overcome the one-size-fits-all model from different approaches, all with a common element: the adaptation of objectives, learning processes and resources for the learning conditions of students.

- In this field, research reports diverse student learning conditions that are tensors for adaptation: learning styles, motivation, student performance, but from the search it appears that thinking processes are not one of these tensors.
- Most adaptive learning systems marginalize the teacher. Finding proposals that go against this trend is important to generate a proposal that gives an active role to the teacher and contributes to the adaptation process.
- When it comes to curriculum design, it is apparent that it does not have the same investigative and innovative intensity today as with the field of adaptive learning. However, concerns can be identified regarding the design of more relevant curricula for the needs and forms of education in today's world.
- Nevertheless, there has been some progress in curriculum design, especially in involving teachers in curriculum design, in the creation of study materials and in their use. At least three paradigms have gone beyond the instrumental (classical) paradigm: the communicative paradigm, the artistic paradigm, and the pragmatic paradigm.
- Another central theoretical field in this thesis is mathematical thinking. In this field the emergence of diverse models is noted, some more developed than others, centered all in the processes of mathematical thinking, leaving aside the learning of specific content.
- In relation to the development of mathematical thinking, the APOS theory proposes the need for students to generate schemes that allow them to face non-routine or challenging problems. In this doctoral thesis this will be a fundamental axis for the design of the activities.
- Also, from APOS theory, but this time in relation to curriculum design, the work that has been done in terms of the genetic decomposition of mathematical domains will serve as a reference for this research.

- With few exceptions, the intensity of research in the field of teaching-learning of linear algebra has decreased in recent years, compared with the level it had at the end of the last century. However, it remains a valid field of research in which many questions remain unresolved.

CHAPTER 2. THEORETICAL FRAMEWORK

As will be seen in the third chapter, one of the phases of the research method chosen for this doctoral work, design-based research, is the development of solutions within an appropriate theoretical framework. For this reason, the theoretical framework described in this chapter is focused on providing the necessary elements for carrying out the design proposed in the objective of this research. This framework has three fundamental axes: adaptive education, design of the curriculum, and development of mathematical thinking, which form the tripod on which this research is based.

2.1 Adaptive education

The educational model proposed to address the needs of the first industrial revolution at the end of the eighteenth and beginning of the nineteenth century, which established education for large groups of people to feed the new productive needs of the market that was uniformly structured, began to be questioned in the middle of the twentieth century, because students begin a learning process possessing a wide variety of abilities, skills, knowledge, attitudes and values.

One of the first to question this model was Lee Cronbach, who in 1957 theorized that learning outcomes are based on the interaction between the attributes of the individual and the variables of the educational process and thus promoted the differentiation of instruction according to people's cognitive abilities (Murray & Pérez, 2015). Later he realized that there could be other variables for adaptation and in 1975 he included the factors of cognition and personality.

Also, in the late 1970s, two former presidents of Division 15 of the American Psychological Association (APA), Robert Glaser (1977) and Richard Snow (1980), each proposed using adaptive education to solve the problem of mass group teaching (Corno, 2008).

These first proposals did not have much success; however, with the arrival of computer systems, the concept of adaptive education was taken up again, and technological developments to be implemented with students were proposed. These developments have been structured at different levels and with different characteristics. Below is a diagram summarizing these ideas.

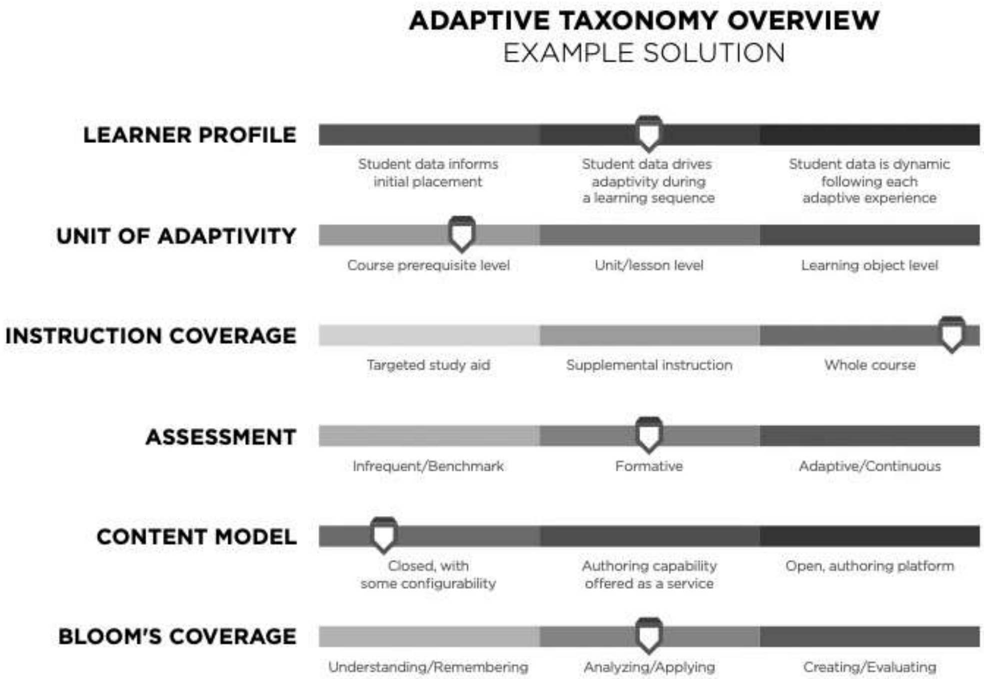

Figure 6. Adaptive taxonomy solution overview (sample solution).

Newman (2013).[80]

According to Newman (2013), the taxonomic categories in this diagram are described as follows:

- *"Learner Profile (or student model) is a structured repository of information about the learner used to inform and personalize the learning experience.*
- *Unit of Adaptivity refers to the structure of the instructional content and the scale at which that content is modified for specific learner needs.*

[80] Newman, A. et al. (2013). Learning to adapt: Understanding the adaptive learning supplier landscape. Education Growth Advisors. p. 7.

- *Instruction Coverage refers to the pedagogical flexibility of a product to deliver an adaptive learning experience and the scope/scale of that experience within the context of a course.*
- *Assessment is the frequency, format, and conditions under which learners are evaluated.*
- *Content Model (or domain model) describes the accessibility of the product's authoring environment to instructors or other users and their ability to add and/or manipulate instructional content in the system.*
- *Bloom's Coverage highlights to what extent a product can support the learning objectives within the Cognitive Domain of Bloom's Taxonomy".*[81]

These taxonomic categories will be considered in the design of an education system based on an adaptive curriculum.

In general, the principles of adaptive education emphasize the fact that the goal of this paradigm is to achieve a common learning goal with students although their individual conditions, such as prior performance, aptitude, or learning styles differ (Ikwumelu, 2015).

So, in summary, the term adaptive learning refers to a method of instruction that allows students to live personalized learning experiences based on the design of a curriculum that is not only personalized, but also adapts to different moments of learning, generating personalized learning trajectories derived from algorithms, course analysis, evaluation data, and student feedback.

A key feature of adaptive education systems is formative assessment. Whether done on an ongoing basis, or at specific times in the learning process, this assessment updates the student model and allows decisions to be made about the path of the domain model– it is not part of the taxonomy. A recurrent feature of current models of adaptive education is the use of formative assessment in

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[81] Newman, A. et al. (2013). Learning to adapt: Understanding the adaptive learning supplier landscape. Education Growth Advisors. p. 7.

terms of performance measurement, considering as an indicator the outcome of the student's responses to certain questions. From the point of view of the development of thinking, this has a disadvantage, since several students could arrive at the same response using different thought paths, which the system is unable to detect. In this sense, the proposal of the present research is to implement evaluation strategies that allow the characterization of the ways students are thinking while facing problem situations, and do not only measure if their answers are correct or not (Durlach, 2019).

2.1.1 Kinds of adaptative educational environments

According to Brusilovsky (2000), the first adaptive learning environments were intelligent tutoring systems (ITS). These systems provided the student with almost no learning material but assumed that the necessary knowledge was acquired outside the system. These systems were dedicated to making adaptations of the presentation.

Later, the systems began to integrate capabilities to guide the navigation of users, which led to hypermedia adaptive systems. This work on adaptive navigation was also influenced by research on the sequencing of curricula, and with it the student began to be provided with the sequence of units of knowledge and activities most suited to his/her conditions.

Adaptive systems have solved the problem of customization of education in two major ways; on the one hand, the creation of virtual learning environments that meet specific design requirements, and on the other hand, the creation of plugins or changing properties of some open-source Learning Management Systems (LMS) modules, such as Moodle (Múdrak, 2018). These systems do not have adaptive features *per se*, however they have been used in numerous research programs as containers of student, domain and adaptation models, in addition to being used in most teaching processes that have some virtual component. A learning management system manages the contents and activities of a learning process and enables an effective relationship between students and tutors.

Regarding the adaptation of the interface of adaptive education systems, two concepts are related: adaptability and adaptivity. *"Adaptive systems dynamically change their features to support the users constantly in their activity, while adaptable systems only provide the user with various adaptation mechanisms for this change"*.[82]

2.1.2 Factors on which adaptation depends (learning conditions)

As mentioned above, modeling the factors on which student learning depends is fundamental to the development of an adaptive educational system. However, these factors are not obvious, not only because there is a great variety, but also because research is ongoing as to which factors are truly responsible for the quality of learning.

The first factor to consider for the generation of adaptive curricula is cognitive structure. Research has shown that cognitive structure has a great impact on learning outcomes. "Cognitive structure describes the qualitative development of knowledge and contains two parts: the knowledge level of learners and the knowledge structure (e.g., prerequisite relationships) of the learning elements. The knowledge level reflects the mastery of learning elements that are still evolving and cannot be directly observed, while the knowledge structure captures the cognitive relationships constructed by the learner between learning elements" at a given point in time. The knowledge level of learners can be diagnosed with an a priori assessment of the learning process. The design of this assessment implies the correct elaboration of the content or domain model. On the other hand, the structure of knowledge is represented in the development of learners' thought processes.

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[82] Mudrák, M. (2018). Personalized e-course implementation in university environment. *International Journal of Information and Communication Technologies in Education, 7*(2), p. 18.

Another adaptive factor to consider in this research is learning styles. According to Keefe, "learning styles are the cognitive, affective, and physiological traits that serve as relatively stable indicators of how students perceive interactions and respond to their learning environments."

Research findings report many learning styles based on different conditions: how information is acquired, how information is processed, and how information is used. It is important to understand that these three types of classifications of learning styles are not disjoint, but have elements in common, because there is a lot of relationship between ways of acquiring information, ways of processing information, and ways of using information. There are also models for characterizing learning styles based on personality dimensions, emotional preferences, and social preferences.

On the other hand, models of learning styles can be classified, according to Coffield et al (2004b), as quoted in Graf (2007), into four families: *"The first family is based on the idea that learning styles and preferences are largely based on the constitution, including the four modalities: visual, auditory, kinesthetic and tactile. The second family deals with the idea that learning styles reflect deeply rooted characteristics of the cognitive structure, including patterns of skills. A third category refers to learning styles as a component of a relatively stable personality type. In the fourth family, learning styles are considered flexible and stable learning preferences".*[83]

Table 2. Summary of examples of learning style models.

Graf, S. (2007).[84]

Learning styles as relatively stable personality type	Learning styles related to approaches and strategies	Constitutionally-based learning styles	'Flexibly stable' learning styles
Myers-Briggs	Pask	Dunn and Dunn	Kolb
	Entwistle	Gregorc	Honey and Mumford
	Grasha-Riechmann		Herrmann
			Felder and Silverman

[83] Graf, S. (2007). Adaptivity in learning management systems focussing on learning styles. Vienna: Vienna University of Technology. p. 6.
[84] Graf, S. (2007). Adaptivity in learning management systems focussing on learning styles. Vienna: Vienna University of Technology. p. 6.

The examples of learning style models proposed by Graf (2007) do not include the model that will be used to make the adaptation in this research, which was proposed by Sierpinska (2000) in her article "On Some Aspects of Students' Thinking in Linear Algebra". In this article, she describes three modes of thinking, which could be considered as an example of learning styles, according to the definition previously established for these. Indeed, since the modes of thinking are associated with deep cognitive characteristics, they could be said to be part of the second family of learning styles.

Although these modes of thinking have been previously described in this paper, it is important to establish their definitions in this theoretical framework, as follows:

- The synthetic-geometric mode of thinking uses the language of geometric figures - planes and lines, intersections - as well as their conventional graphical representations.
- The analytic-arithmetic mode of thinking represents mathematical objects in terms of their formal relations expressed in numerical or algebraic form.
- The analytic-structural mode of thinking abstracts the algebraic and geometric elements into structural elements, i.e., general properties of these elements, which, in a way, erase the 'form' and leave only what is essential.

Below is a table showing the representation of some of mathematical objects related to vectors in each of the modes of thinking.

Table 3. Summary of features of adaptive instructional systems.
Author's own elaboration.

Object	Synthetic-geometric	Analytic-arithmetic	Analytic-structural
Vectors	Arrows on a straight line, plane or in space, with coordinates.	'Boxes' with organized numbers.	You do not see components within vectors, you see vectors, units, as elements of a space.
Vector operations	Movements, deformations, and compositions of arrows.	Operations with the numbers in the 'boxes'.	Formally defined operations, with abstract properties.
Norm	Arrow length.	A number that is calculated with a certain algorithm (larger components give a larger number). It is the operational definition of the (Euclidean) distance function.	Any function that associates an element of the vector space with a real number, which satisfies 'certain' properties.
Scalar product	Position of an arrow with respect to the other arrow.	An algorithm that associates a real number with two vectors.	A function that is defined from the Cartesian product of the space itself in the base body and that satisfies properties that are independent of the space and independent of the body.

To make the adaptation in the present research, only the first two modes of thinking were considered, namely: synthetic-geometric and analytic-arithmetic, since students in a first course of linear algebra, at least current Colombian students, do not yet have the analytic-structural mode of thinking develop. However, in terms of the development of mathematical thinking, this adaptation did not imply keeping the student in the same mode of thinking during the whole learning process, but encouraging the change of representation and, in effect, trying to get the student to adopt the analytic-structural mode of thinking.

2.1.3 Structure of an adaptive educational system

An adaptive learning environment model can be subdivided into a learner (or student) model, domain model and adaptation model (Murray & Pérez, 2015).

- Student model: this model contains the parameters that describe the characteristics related to the students' ways of learning. This model can be defined before the instruction process begins, or it can be adapted dynamically. A complete student model includes attributes of learning (Popescu, 2010), mechanisms to diagnose and infer learner characteristics (Brusilovsky & Peylo, 2003) and mechanisms for assessing student performance (Lee & Park, 2008). In particular, the *"user model must contain important information about the user such as domain knowledge, learning performance, interests, preference, goal, tasks, background, personal traits (learning style, aptitude...), learning activities, environment (context of work) and other useful features".*[85]
- Domain model: this model represents and organizes the knowledge to be learned by the student. This organization is not only thematic in content, but also at the level of resources such as purposes, tasks, activities, assessment, and other elements that are used in learning. This is the model in which all elements of the curriculum are found. *"The domain model is implemented as a repository of learning resources described by a set of metadata. The metadata contains various traits, including physical characteristics (media type, format, location, etc.), knowledge characteristics (knowledge type, difficulty level, etc.) instructional role (such as defined in Bloom's taxonomy), and relationship specifications (hierarchical, peer, etc.)".*[86]
- Adaptation model: this model is responsible for producing the articulation between the student model and the content model, determining what, how and when it should be

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[85] Fröschl, C. et al. (2008). Learner Model in Adaptive Learning. *World Academy of Science, Engineering and Technology* (21). p. 7.
[86] Murray, M. C., & Pérez, J. (2015). Informing and Performing: A Study Comparing Adaptive Learning to Traditional Learning. Informing Science: The International Journal of an Emerging Transdicipline (18), 111. p. 115.

adapted. Adaptation can occur through fixed rules or through machine learning, among other mechanisms.

In summary, a specific characterization of the student and its implementation in the form of a model, added to the elaboration of a model that correctly interprets the curricular elements, are the input for the design of an adaptation model that allows adjusting teaching mechanisms to the specific conditions of learning.

In a second direction, there are two approaches in personalized e-learning: static and dynamic. If a questionnaire is applied at the beginning of the course to determine the characteristics of the students (learning style, previous knowledge, motivation, etc.), to adapt it to these characteristics, the static approach is being used. But if, in addition to this or in its place, the characteristics of the course are adapted during its execution from the observation, in specific moments or in real time, of the incidences of the learning process, the dynamic approach is being used (Karagiannis & Satratzemi, 2016).

2.2 Curriculum design

Enrolled in the instrumental paradigm, Tyler (1971) considers the curriculum as a document that foresees the ends and results of learning, establishing educational experiences and the appropriate pedagogical practice to achieve them.

Going further, but still in the same instrumental paradigm, Taba (1974) proposes a theory of curriculum development that contains the following steps:

1. Diagnosis of needs
2. Formulation of objectives
3. Content selection
4. Organization of contents

5. Selection and organization of learning activities
6. Determination of what is to be evaluated and the ways and means to do so.

This curriculum design paradigm is well suited to the specificity of adaptive curricula because the diagnosis of needs (and of possibilities according to the factors of adaptation), whether static or dynamic, makes it possible to adapt the characteristics of the student model to those of the domain model. Besides the organization of content and the selection of learning and assessment activities can be done by one of the two curricular design models that were mentioned in the chapter on the state of the art, since both are highly adequate for supporting adaptive education processes: the method of theoretical analysis of the activity and the method of analysis of tasks.

It is convenient to remember the definition of these two methods. On the one hand, as mentioned earlier in that chapter, the method of theoretical analysis of the activity for curriculum design proposes that the curriculum be designed based on the needs of the students' incoming profile and the possibilities with which they enter the program (Hernández Díaz, 2018).

And, on the other hand, the task analysis method consists in developing hierarchies of simple objectives (simple tasks) that facilitate the learning of higher objectives (more complex tasks). This method implies the identification of concepts included in the curriculum and the establishment of a hierarchy of prerequisites among them to allow the student to reach the acquisition of competence. The sequencing of tasks is carried out based on various criteria that are not only intrinsic to the content, but also come from the characteristics of the students (Resnick, 1973).

Independently of these methods, or better yet, in articulation with these methods, collaborative paradigms for curriculum design can be considered. In the book by Pieters, J., J. Voogt, & N. Pareja Roblin (2019) three paradigms of this type are presented which change the perspective of teacher participation in curriculum design.

These three paradigms are briefly described below.

The first is the communicative paradigm, rooted in descriptive theories about curriculum design that *"study what people actually do when they design curricula and are particularly concerned with how people arrive at answers"*.[87] This paradigm seeks to reach a consensus based on the knowledge, beliefs, and values of the main design stakeholders, and for this purpose teachers participate by collaborating as a team.

The second is the artistic paradigm, rooted in the theory of situated cognition that *"claims that knowing is rooted in social activities, context and culture"*.[88] The principle of this paradigm is the individual process of construction of meaning, often based on the expertise and experience of the connoisseur (Visscher-Voerman & Gustafson, 2004). It would seem that this paradigm is very similar to the instrumental paradigm, however, the two differ because in the instrumental paradigm curriculum design is conceived as a linear process, but in the artistic paradigm this process is conceived as an open-ended process in which means and ends are interdependent.

The third is the pragmatic paradigm whose basic interest is whether the design works in practice and is found useful by the end users (Visscher-Voerman & Gustafson, 2004). This paradigm emerged from the practice of software engineering. In this paradigm, curriculum design is based on the development of prototypes and their implementation to develop design principles and continuous improvement. It is a highly iterative process (Pieters & Pareja Roblin, 2019).

The three paradigms presented are opposed to the instrumental paradigm in providing active participation for teachers in curriculum design, since in the instrumental paradigm, rooted in

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[87] Pieters, J. V., & Pareja Roblin, N. (2019). Collaborative curriculum design for sustainable innovation and teacher learning. Springer Nature. p. 8.
[88] Pieters, J. V., & Pareja Roblin, N. (2019). Collaborative curriculum design for sustainable innovation and teacher learning. Springer Nature. p. 9.

prescriptive theories (which seek to create the best possible curriculum), *"teachers are seen above all as implementers of curriculum*.[89]

In the following chapter concerning research methodology it will be argued and decided which of the paradigms will be used for this research. However, independent of which one should be chosen, it will be:

- Oriented to the development of mathematical thinking, and not only to the learning of contents.
- Adjusted to the diverse characteristics of the types of mathematical thinking.
- Built around problem solving as a central axis.

2.3 Development of mathematical thinking

The development of mathematical thinking is a central purpose of the framework for curriculum design sought after in this research.

2.3.1 Concept of mathematical thinking

Indeed, Drijvers et al (2019) say that *"The importance of mathematical thinking as a key higher order learning goal in mathematics education is widely accepted"* [90], showing the importance of going beyond the teaching of mathematical concepts and algorithms to the development of mathematical thinking and reinforcing this statement by agreeing with Devlin (2012) that *"The danger of not making this shift, however, is that mathematics in school differs drastically from mathematics in a professional or academic setting"*.[91]

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[89] Pieters, J. V., & Pareja Roblin, N. (2019). Collaborative curriculum design for sustainable innovation and teacher learning. Springer Nature. p. 8.
[90] Drijvers, P. et al. (2019). Netherlands, Assessing mathematical thinking as part of curriculum reform in the. *Educational Studies in Mathematics*. p. 3.
[91] Drijvers, P. et al. (2019). Netherlands, Assessing mathematical thinking as part of curriculum reform in the. *Educational Studies in Mathematics*. p. 3.

Of course, there is no single definition of the concept of mathematical thinking. Cantoral (2005) says *"we will use the term mathematical thinking to refer to the ways in which people professionally engaged in mathematics think"*.[92] He also says that, when speaking of mathematical thinking it must be done from a perspective *"properly in the sense of mathematical activity as a special form of human activity. So, we must be interested in understanding the reasons, procedures, explanations, scripts, or verbal formulations that the student constructs to respond to a mathematical task, just as we are concerned with deciphering the mechanisms by which culture and the environment contribute to the formation of mathematical thoughts"*.[93]

In a tautological way like Cantoral, Olive Chapman (2011) has described mathematical thinking as the type of thought put into play when doing mathematics.

On the other hand, the book of Mason, Burton and Stacey (1982, 2013), "Thinking Mathematically", begins the introduction of the first edition with the following sentence: *"Thinking Mathematically is about mathematical processes, and not about any particular branch of mathematics"*.[94] This is the meaning that will be given to mathematical thinking in the present research, but it is necessary to specify that, although there are general processes of mathematical thinking that are involved in all branches of mathematics, there are other processes that are more specific to some branches than others. This idea will be used in the framework for the design of curricula oriented to the development of mathematical thinking, so that although this framework will serve to design curricula oriented to the teaching of any branch of mathematics, a modular component will be used that contains the most appropriate thinking processes for the subject that the specific curriculum is intended to design.

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[92] Cantoral, R. et al. (2005). *Desarrollo del pensamiento matemático.* México: Trillas. p. 18.
[93] Cantoral, R. et al. (2005). *Desarrollo del pensamiento matemático.* México: Trillas. p. 18.
[94] Mason, J., Burton, L., & Stacey, K. (1982). *Thinking Mathematically.* London: Addison Wesley. p. viii.

In his description of mathematical thinking, Tall (2013) uses the concept of *procept* with which he seeks to encapsulate the two senses, concept and process, of a mathematical symbol. In this regard, he says: "*An elementary procept is the amalgam of three components: a process that produces a mathematical object, and a symbol that is used to represent either process or object. A procept consists of a collection of elementary procepts having the same object".*[95] He also emphasizes the difference between elementary mathematical thinking and advanced mathematical thinking and makes it clear that *"the move from elementary to advanced mathematical thinking involves a significant transition: that from describing to defining, from convincing to proving in a logical manner based on those definitions. This transition requires a cognitive reconstruction which is seen during the university students' initial struggle with formal abstractions as they tackle the first year of university. It is the transition from the coherence of elementary mathematics to the consequence of advanced mathematics, based on abstract entities which the individual must construct through deductions from formal definitions".*[96] This distinction is important in the present doctoral thesis because the field of study it addresses is university mathematics teaching, which touches more closely on advanced mathematical thinking and must consider the consequences of the transition of students' minds toward abstraction and deduction.

For their part, Drijvers et al (2019) argue that mathematical thinking can be described from three approaches.

The first one is problem solving, conceived *as "tasks that are non-routine to the student and invite to think of a possible solution strategy".*[97] Facing the student with problems under this conception causes the development of thinking processes and skills for the use of previous knowledge and experiences.

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[95] Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*. p. 121.
[96] Tall, D. (1991). *Advanced mathematical thinking.* Springer Science & Business Media. p. 20.
[97] Drijvers, P. et al. (2019). Netherlands, Assessing mathematical thinking as part of curriculum reform in the. *Educational Studies in Mathematics*. p. 3.

The second approach is modeling. Modeling involves three processes: the connection of mathematics with the world around us, its application, and the invention of new mathematics to solve problems. This approach is closely related to the first one since modeling could be considered one of the processes for solving problems.

The third approach is given in terms of abstraction, conceived as *"an activity by which we become aware of similarities [...] among our experiences"*.[98] Drijvers et al (2019) warn that this component has been underestimated in the teaching of mathematical thinking.

2.3.2 Frameworks of development of mathematical thinking

According to Scheiner & Pinto (2019): *"Three main traditions have framed recent discussions on cognitive processes underlying mathematical knowing and learning: activity theory viewpoints, process-object perspectives, and situated-knowledge considerations"*.[99] The first of these traditions considers that mathematical thinking is a human activity and, in that sense, people build a mathematical construct from the relationships between other already known mathematical constructs. The second tradition considers that it is from cognitive processes that structural concepts are formed, following an operative process. The third tradition argues against reductionist and mentalist narratives when it comes to recognizing the situation of mathematical comprehension. It proposes that mathematical thinking is a complex construct (e.g. dynamic, recursive, multiple interactions and levels).

This doctoral research is framed in the second of these traditions, since it seeks that students generate schemes that allow them to face structural situations, i.e., challenging or non-routine problems, which do not only imply the repetition of procedures, but the construction of thought.

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[98] Skemp, R. R. (1986). *The psychology of learning mathematics.* Harmondsworth, U.K.: Penguin. p. 21.

[99] Scheiner, T., & Pinto, M. M. (2019). Emerging perspectives in mathematical cognition: contextualizing, complementizing, and complexifying. *Educational Studies in Mathematics*, *101*(3). p. 357.

According to APOS theory proposed by Dubinsky (1991), mathematical concepts are described in terms of a genetic decomposition into their constituent actions, processes and objects in the order the learner should experience them so that he or she will be able to construct understanding (Stewart & Thomas, 2009). According to this theory, an action is a transformation of objects which the individual can carry out step by step, obeying external stimuli. When the individual reflects on these actions, he can internalize them and they become processes, since the same transformations can be carried out in the mind of the individual, without the need for external stimuli. When there is a need to apply actions on processes, they are encapsulated to give rise to objects (Oktaç & Trigueros, 2010). A schema for a mathematical concept is a structure which is constructed by building relations and transformations among several actions, processes, objects and other, previously constructed, schemas.

In APOS theory, schemas are evoked by individuals in the solution of problem situations. Schemas evolve continuously by the mechanisms of assimilation and accommodation. The development of a schema has three levels: Intra-, Inter- and Trans-. The intra- level is characterized by superficial relations among components of the schema. In the inter- level, there is awareness of transformation relations between the elements of the schema, and in the trans- level these relations are developed and the schema is coherent in the sense that it is possible for the individual to distinguish those problems where the schema can be applied (Trigueros, 2018).

A separate space is reserved for the concept of genetic decomposition. *"A genetic decomposition is a hypothetical model that describes the mental structures and mechanisms that a student might need to construct in order to learn a specific mathematical concept"*.[100] A genetic decomposition is a hypothesis based on the learning

[100] Arnon, I. et al. (2014). APOS theory. A Framework for Research and Curriculum Development in Mathematics Education. New York: Springer. p. 27.

experiences and mathematical and didactic knowledge of the researcher or teacher. A genetic decomposition is described in terms of the actions, processes, objects and schemes that the student needs and is forming in the process of development of mathematical thinking.

In their book, Mason, Burton and Stacey (1982) propose a method to improve mathematical thinking, as well: *"tackling questions conscientiously; reflecting on this experience; linking feelings with action; studying the process of resolving problems; and noticing how what you learn fits in with your own experience"*.[101] They also make five assumptions about mathematical thinking with which this research fully agrees and which will be used as the framework for the design of the solution to the problem addressed by this thesis: *"You can think mathematically; mathematical thinking can be improved by practice with reflection; mathematical thinking is provoked by contradiction, tension and surprise; mathematical thinking is supported by an atmosphere of questioning, challenging and reflecting; mathematical thinking helps in understanding yourself and the world"*.[102] These assumptions are the reason why challenging problems are an essential element in the design of curricula based on the framework developed with this thesis.

2.3.3 Characterization of challenging problems

For the creation of non-routine or challenging problems, an inductive analysis was made of the tests of the Colombian Mathematical Olympiads of the last four years. From this analysis it was concluded that there are at least five types of challenging problems, namely:

- Problems where common calculations are needed, but for special objects or constraints.
- Problems where it is necessary to find objects with special characteristics.

[101] Mason, J., Burton, L., & Stacey, K. (1982). Thinking Mathematically. London: Addison Wesley. p. viii.
[102] Mason, J., Burton, L., & Stacey, K. (1982). Thinking Mathematically. London: Addison Wesley. p. x.

- Problems where the question is reversed, i.e., from a solution the student is asked to propose a question.
- Problems where examples or counterexamples are requested for certain mathematical results.
- Problems asking for full or partial proofs of mathematical results.

2.3.4 Mathematical thinking processes in linear algebra

Research, such as that of the Linear Algebra Curriculum Study Group, has related the problem that linear algebra is a subject that must be taught to a large diversity of students and, in that sense, must be adapted to the particular characteristics of those students. Indeed, this group proposes a set of recommendations to address this situation:

"1. The syllabus and presentation of the first course in linear algebra must respond to the needs of client disciplines.

2. Mathematics departments should seriously consider making their first course in linear algebra a matrix-oriented course.

3. Faculty should consider the needs and interests of students as learners.

4. Faculty should be encouraged to utilize technology in the first linear algebra course.

5. At least one 'second course' in matrix theory/linear algebra should be a high priority for every mathematics curriculum".[103]

Dubinsky (1997) identifies difficulties in teaching linear algebra. These difficulties are: the teaching by example and not through the construction of objects by students; the weak understanding of the

[103] Carlson, D. et al (1993). The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra. *The College Mathematics Journal, 24*(1). p. 41.

central concepts of mathematics that are used for the teaching of linear algebra; the lack of interaction of students with problems in which they can actually apply, not routinely, the objects they have been able to construct.

To overcome these difficulties, Dubinsky proposes to use APOS theory. This constructivist theory is based on the motto proposed by Piaget: *"to know something is to transform it", so that emphasis is placed on two elements: transformations (actions and processes) and what is transformed by them (objects and schemes). Successive thought processes make actions and processes evolve towards objects and schemes: "actions are constructed by repeated responses to stimuli; processes are constructed either by interiorizing actions or by transforming existing processes; objects are constructed by encapsulating processes; and, in de-encapsulating an object, the only processes an individual can obtain are processes which were encapsulated to construct this object".*[104]

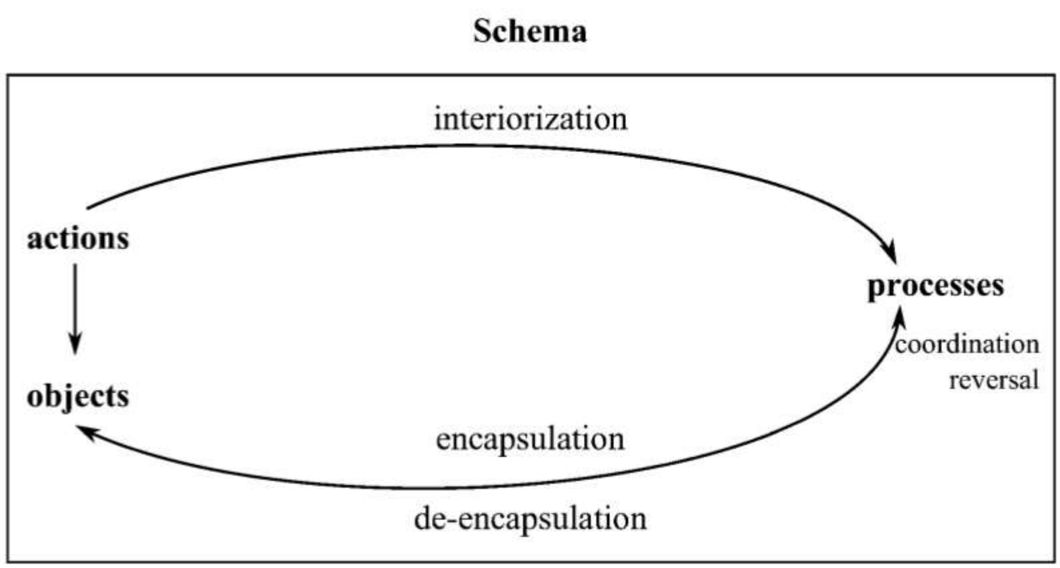

Figure 7. Mental constructions in APOS theory.

Dubinsky, (1997).[105]

[104] Dubinsky, E. (1997). Some thoughts on a first course in linear algebra at the college level. *MAA NOTES*. p. 16.
[105] Dubinsky, E. (1997). Some thoughts on a first course in linear algebra at the college level. *MAA NOTES*. p. 16.

Chapter conclusions

The theoretical framework is fundamental to the research method chosen for the development of this doctoral work. Therefore, the conclusions of this chapter will determine the theories that will support the design and implementation of the solution to the proposed problem.

- In relation to adaptive education, Newman's taxonomy (2013) will be used to properly tune the characteristics of adaptation for curriculum design. The curriculum will be implemented on a adaptable learning management system (ALMS) built specifically for this research. In the design of the ALMS, a structure containing three models, the student, domain and adaptation models, will be used. The adaptation model chosen among all the possible existing learning conditions to adapt the teaching processes was that of the modes of thinking proposed by Sierpinska (2000). It is also important to specify that the adaptive model will be static in nature.
- In relation to the development of thinking, the framework proposed by the APOS theory will be used to analyze the students' progress, in an integrated manner with the modes of thinking, and genetic decomposition (APOS) will be used to structure the domain model.

A more specific list of the theoretical frameworks selected for this research can be found in the following chapter.

CHAPTER 3. RESEARCH METHODOLOGY

This chapter outlines and describes the methodology used in the present research.

3.1 Research approach

Based on the general objective and the specific objectives proposed for the present research, qualitative is research approach most appropriate for structuring it and carrying it out. As Gómez & Roquet (2012) say: *"Qualitative methodologies are oriented towards the understanding of unique and particular situations, they focus on the search for meaning and sense given to the facts by the agents themselves, and on how certain phenomena or experiences are lived and experienced by the individuals or social groups under investigation"*.[106]

The following describes the design and implementation process of an adaptive curriculum oriented to the development of mathematical thinking, specifically instantiated in the teaching-learning of linear algebra.

3.2 Method of investigation

As the present research aims to design a solution to a problem situation, implement it, evaluate it, and refine it, the most relevant research method for this purpose is design-based research.

This method, which has recently emerged in research theory, was initially proposed by Brown and Collins (1992), and arises from the need to link research, educational design and innovation, determining how, when and why educational innovations work in practice.

Cobb et al (2003) give the following definition for the method of design-based research. *"Prototypically, the design of experiments involves both the engineering of particular forms of learning and the systematic study of those forms of learning in the context defined by the means that support them. This designed*

[106] Gómez, S., & Roquet, J. V. (2012). *Metodología de la investigación.* México: Red tercer milenio. p. 47.

context is the object of tests and revisions, and the successive iterations that result, play a role like the systematic variation in experiments".[107]

Some features that are relevant to design-based research are (Design-Based Research Collective, 2003):

- The main objectives of the design of learning environments and the development of learning theories or "proto-theories" are interconnected.
- Development and research occur through continuous cycles of design, development, analysis, and redesign.
- It should explain how designs work in real scenarios.
- It is based on intense collaboration between researchers and stakeholders.
- It involves a commitment to building and explaining theories while solving real problems.

The following figure presents an outline for the development of this research method:

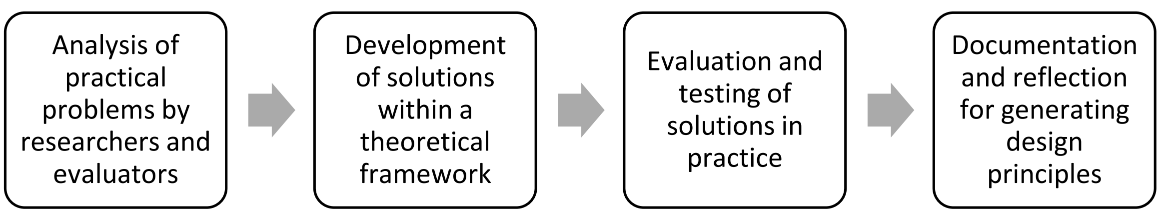

Figure 8. Design-Based Research Process.

Gómez & Roquet (2012).[108]

In the processes presented in the previous figure it is necessary to emphasize that, given the characteristics of a doctoral research project, it is necessary that the last process not be restricted only to the generation of design principles, but also that it contribute new theoretical elements to the conceptual field that is being developed.

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[107] Cobb, P. et al. (2003). Design Experiments in Educational Research. *Educational Research, 1*(32). p. 9.
[108] Gómez, S., & Roquet, J. V. (2012). *Metodología de la investigación.* México: Red tercer milenio. p. 68.

From the field of research in mathematics education there is a paradigm closely related to research based on design: didactic engineering. In fact, in the conference presented by Godino et al (2013) at CERME, a relationship of complementarity is established between didactic engineering, of French origin (Artigue, 1989), and the design-based research current, of Anglo-Saxon origin. In the work of Godino et al (2013) it is stated that the *"problem addressed in both approaches is the design and evaluation of educational interventions, which provide research-based resources for the improvement of mathematics teaching and learning"* [109] and it is concluded that didactic engineering can be conceived as a specific case of design-based research, since the first one is fundamentally linked to the theory of didactic situations, while the second one can adopt other theoretical frameworks.

3.3 Experimental research design

Based on the general structure of the design-based research method, the present research was carried out with the following experimental design.

3.3.1 Analysis of practical problem by researcher

3.3.1.1 Characterization of existing problems in adaptive education

As stated earlier in this paper, adaptive education is a relatively recent solution in which great strides have been made, but which has also experienced sensitive problems.

Thus, the first thing that was done to design the experiment was to characterize those problems, namely:

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[109] Godino, J. D. (2013). La ingeniería didáctica como investigación basada en el diseño. *CERME*. p. 1.

- Most adaptive curriculum designs were initially oriented to the design and implementation of intelligent tutoring systems and then hypermedia adaptive systems, but without considering existing curriculum design criteria and methodologies.
- These adaptive curriculum designs were developed by systems engineers and other related professions, but not by educators or pedagogues, so that educational theories were not integrated in most of them.
- The curricular designs developed were based on student performance and not on the process, which resulted in students with different processes but with the same performance results being classified in the same group.
- In general, the aim was to marginalize the teacher from the educational process, replacing him or her with learning management systems.
- There are not many records of group work as a tool to influence the results of the adaptive educational process.

3.3.1.2 Decision of the subject to implement the solution

In line with the justification of this doctoral thesis, in which it is accepted that adaptive education has as one of its main advantages that of working with groups of students with diverse characteristics in terms of knowledge levels and structure, interests, learning styles and other factors, the implementation of the present research will be done within the framework of the subject linear algebra, whose students come from different academic programs, specifically, engineering and mathematics programs.

On the other hand, in line this time with the ulterior purpose for which this adaptive curriculum is designed, namely, the development of mathematical thinking, it is important to state that the thematic contents chosen are important, but they are pretexts and are at the service of such ulterior purpose.

3.3.2 Development of solutions within a theoretical framework (first iteration)

3.3.2.1 Determination of theoretical frameworks to use

To propose an adequate solution to the research problem, theoretical frameworks with coherent characteristics were selected. These frameworks were proposed in the three categories that have been developed in the chapters on the state of the art and the theoretical framework.

3.3.2.1.1 Curriculum design

The communicative paradigm framed in the pragmatic design of the curriculum was used because is focused on being useful by the end users and is in line with the design-based research method because it makes use of the development of prototypes and their implementation to produce design principles and continuous improvement.

3.3.2.1.2 Adaptive education

The adaptive solution will be based on three models: the student model, the domain model, and the adaptive engine. The adaptive factor used for the student model will be the modes of thinking proposed by Sierpinska (2000), the domain model will be developed using genetic decomposition. This adaptive factor was chosen, although at first, we had thought of working on learning styles, since it is much more relevant from the point of view of the development of mathematical thinking and with the learning contents of linear algebra.

The adaptive engine has the following form.

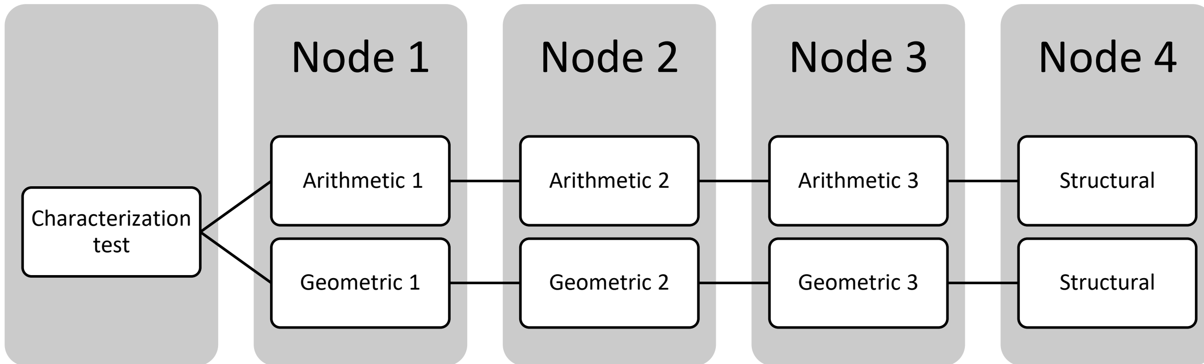


Figure 9. Structure of the adaptive system.

Author's own elaboration.

In the figure above it can be seen that students must first answer the characterization test on modes of thinking. In this test, students are characterized to classify them according to whether they prefer the synthetic-geometric mode of thinking or the analytical-arithmetic mode of thinking. Based on their answers, they will be classified to start at node 1 in one of the two adaptation lines. Node 1 corresponds to the first lesson, the topics of which will be defined from genetic decomposition. The topics of the following levels will be defined in the same way.

Each level of nodes should allow students to understand the concepts and algorithms, exercise them and expand their scope and profundity by solving challenging problems. Each node is divided into the following types of moments:

- Moment with node presentation: includes the objective and contents of the node.
- Moments of theoretical development: the explanation of the theory required to solve the exercises and problems is presented there in text and video form.
- Moments with comprehension questions requiring comprehension: contains questions to be solved by the students. After each of these moments the students find the respective feedback for the questions, with which they will be able to verify their understanding of the theory presented. If they have doubts when verifying their answers, they can ask the teacher.

- Moments with exercises: these are intended to help students become proficient with the algorithms presented in the theory. After each exercise moment, feedback is presented for students to check their results. If they have concerns about the exercises, they can ask the teacher.
- Moments with problems: students will be presented with challenging problems. The feedback on the solution to these problems is provided directly by the teacher to the students. For this purpose, in the teacher's platform there are texts with possible feedback, which help to him/her to better guide the students.
- Moment with node map: there students will find a summary of what they have learned during the node.

A special section is reserved for the characterization test which classifies students according to whether they favor synthetic-geometric or analytic-arithmetic modes of thinking. The design of this test can be found in the next chapter and the complete test can be found in this link: https://cutt.ly/EY4w4x3

The decision was made that the adaptive model would be static in nature, i.e., that students would be classified at the beginning of the process but that no modifications would be made to that classification thereafter. The reason for this decision is that, while a dynamic process would hypothetically be more effective, it would also involve a level of work in the construction of resources that would be more typical of a team than of an individual, in this case the researcher.

3.3.2.1.3 Development of mathematical thinking

The proposal of Drijvers (2019) will be assumed, taking problem solving, modeling and abstraction as the fundamental facets for the development of thinking. In addition, the postulates of APOS theory will be considered for the analysis of the development of students' mathematical thinking and the genetic decomposition of the construct.

A fundamental aspect for the development of mathematical thinking is the proposing of challenging problems. For each node, problems of this type were proposed based on the characterization that can be found in the theoretical framework of this work, which is in accordance with the thesis of Mason, Burton & Stacey (1982): "*You can think mathematically; mathematical thinking can be improved by practice with reflection; mathematical thinking is provoked by contradiction, tension and surprise; mathematical thinking is supported by an atmosphere of questioning, challenging and reflecting; mathematical thinking helps in understanding yourself and the world*".[110] Some examples of challenging problems proposed in the nodes are shown in Appendix 1.

3.3.2.2 Specific curriculum design

3.3.2.2.1 Sequence of themes

Based on the above theoretical frameworks, we proceeded to design the specific curriculum for the linear algebra module related to vectors. Thus, after a decomposition of this construct, it was found that it would be developed through the following thematic content: vectors, projections, orthogonalization and vector spaces, that give rise to each of the four ALMS nodes.

The genetic decomposition of the content for each of the modes of thinking can be seen in Appendix 2.

[110] Mason, J., Burton, L., & Stacey, K. (1982). *Thinking Mathematically.* London: Addison Wesley. p. x.

The following is the thematic sequence, obtained through genetic decomposition. It should be noted that this genetic decomposition depends not only on the subject matter, but also on the modes of thinking.

Table 4. Themes for each node according to modes of thinking.

Author's own elaboration.

Node	Synthetic-geometric mode	Analytical-arithmetic mode
1	- Geometric definition of vector - Arithmetic definition of vector - Operations with vectors: composition (addition) and scaling (multiplication by a scalar) - Magnitude (or norm) of a vector - Direction of a vector - Unit vector - Triangular inequality	- Arithmetic definition of vector - Geometric definition of vector - Operations with vectors: sum and multiplication by a scalar - Dot product (scalar) - Norm (or magnitude) of a vector - Direction of a vector - Unit vector
2	- Dot product (scalar) - Parallel and orthogonal vectors - Projection of one vector onto another - Cauchy-Schwarz Inequality	- Parallel and orthogonal vectors - Projection of one vector onto another - Cauchy-Schwarz Inequality - Triangular inequality
3	- Linear combination - Linear independence - Generator set - Basis and dimension of $\mathbb{R}^n$ - Gram-Schmidt orthonormalization process - Orthogonal basis and orthonormal basis (set)	- Linear combination - Linear independence - Generator set - Basis and dimension of $\mathbb{R}^n$ - Gram-Schmidt orthonormalization process - Orthogonal basis and orthonormal basis (set)

Node 4 is developed in the analytical-structural mode of thinking for both groups of students and has the following thematic sequence:

- Vector space
- Inner product
- Vector subspace

Initially, it was thought to combine the genetic decomposition with the method of theoretical analysis of activity to develop the sequence of learning contents; however, based on a later analysis, it became clear that the genetic decomposition used principles underlying those used in the method of theoretical analysis of activity; therefore, in order to avoid redundancy, only the genetic decomposition was chosen.

While the sequence of topics for each of the modes of thinking differed in the way of presentation and order, the intention remained to have students make representational shifts between modes of thinking during the learning process.

In the first iteration, students were given the following number of problems to solve at each node.

Table 5. Number of problems in each node (first iteration)

Author's own elaboration.

Node	Synthetic-geometric mode	Analytical-arithmetic mode
1	16	32
2	11	16
3	11	11
4*	13	

* For this node, the problems are classified in the analytical-structural mode of thinking.

It is noticeable that nodes in the analytical-arithmetic mode of thinking have more problems, however the workload is similar to the corresponding nodes in the synthetic-geometric mode of thinking, because of the nature of the problems.

3.3.2.2.2 ALMS characteristics

Based on this distribution of topics and the structure of moments of each node, an adaptable learning management system (ALMS) was developed. This system has the following characteristics:

- It is an adaptable and non-adaptive system, as proposed by Mudrák (2018).
- Adaptation is static in nature, i.e., students are classified at first and progress through the system is based on that classification.
- At the moment the student accesses for the first time and after registration, he or she is presented with the mode of thinking characterization test; in subsequent interactions the student enters directly to the nodes.
- The system presents each node moment in the form of a card, and prompts student action in order to advance to the next moment.

- When the student reaches the end of each node, he or she must request authorization from the teacher through the system to continue to the next node.
- The system saves the student's progress in each work session.
- The system has three user profiles: administrator, teacher and student, and for each of them there are different interfaces.
- The teacher interface allows grading the problems, recognizing the progress status of the students and the thinking mode in which they were classified. It also has the feedback of the problems.

The links to access the ALMS is https://adaptive-elearnign-system.web.app/

- To the student interface:
    - Synthetic-geometric mode of thinking: user: estudiante1@gmail,com; password: 123456
    - Analytical-arithmetic mode of thinking: user: estudiante2@gmail,com; password: 123456
- To the teacher interface: user: profesor@gmail.com; password: 123456

3.3.2.2.3 Other pedagogical decisions

To address the difficulties described in the first part of this chapter, the following pedagogical decisions were made for the first iteration of the research:

- Students were required to work individually, except for the resolution of some problems that explicitly asked them to work in groups. The group was to have students of the same mode of thinking.

- When each student solved a problem, he or she had to wait for the teacher's authorization to continue advancing in the system. To give this authorization, the teacher had to grade and give feedback on the solution of the problem.
- The student could communicate with the teacher and the other students through a chat in Teams®.
- The time allowed for progress through the system and completion of the lessons was 20 hours of class time.
- The implementation of this system in this iteration was done virtually (pandemic conditions).

3.3.3 Evaluation and testing of solutions in practice (first iteration)

The first iteration was carried out with students of the subject Linear Algebra of the Fundación Universitaria Konrad Lorenz, Bogotá, Colombia, during the semester 2021-1. This iteration was implemented by the two professors of the subject and directed by the researcher. The 42 students of the course were divided into three class groups. Instructions given to students for the use of the ALMS can be found in Appendix 3.

For the purposes of this study, the designed curriculum was not used for the whole subject, but for a module related to vectors.

The evaluation of the implementation of the curriculum in the linear algebra course was based on the perceptions of teachers and students, and the researcher's analysis of the process.

After the first implementation, a perception survey was applied to students and teachers. Based on this survey, a reflection process was carried out and adjustments were made to the curriculum. This reflection process included the teachers who were in charge of teaching the module based on the designed curriculum. The questions asked in the surveys can be found in https://cutt.ly/IY4w0Ul

3.3.4 Development of solutions within a theoretical framework (second iteration)

3.3.4.1 Changes implemented in the curriculum

Based on the reflection process from the first iteration, the following changes were made to the curriculum.

- Working groups were made up of two or three students, i.e., the work was not individual. This was done so that communication among students would enhance the development of mathematical thinking. The students composing a working group communicated via WhatsApp; in each WhatsApp group the teacher was also included.
- When the students solved the problems, they could continue advancing through the node, so they would have smoother progress and better use of time.
- The number of problems students had to solve at each node was reduced.

Table 6. Number of problems at each node (second iteration)

Author's own elaboration.

Node	Synthetic-geometric mode	Analytical-arithmetic mode
1	14	18
2	7	12
3	9	9
4*	7	

* For this node, the problems are classified in the analytical-structural mode of thinking.

3.3.4.2 Changes implemented in the ALMS

Corresponding changes were made to the ALMS.

- The system shows a progress bar for each node.
- The number of problems to be solved was added at the time of node presentation.
- In addition to the explanations in text format provided during the moments of theoretical development moments, video explanations were added.

3.3.5 Evaluation and testing of solutions in practice (second iteration)

The second iteration was carried out with students of the subject Problems Solving of the Universidad Antonio Nariño, Bogotá, Colombia, during the semester 2021-2. This iteration was implemented by the researcher. The 70 students of the course were divided into two class groups. Once again, the designed curriculum was applied in one of the modules of the course. Instructions given to students for the use of the ALMS can be found in Appendix 3.

Also, after this implementation a perception survey was applied, this time only to students. The survey had the same questions that were applied to the students in the first iteration.

The results of the reflection process from the two iterations are presented in the following section.

3.3.6 Documentation and reflection for generating design principles

Finally, the design principles of an adaptive curriculum for the development of thinking were generated**,** thus satisfying the overarching objective of this research work. These principles were not proposed exclusively for the subject of linear algebra, but were intended to have a modular logic, so that they can be used in other subjects. These principles are also presented in the following section.

Chapter conclusions

- The methodological design presented in this chapter follows the logic of the design-based research method; thus, two iterations of the implementation were projected, so that the instruments could be improved in the second iteration with respect to the first.
- On the other hand, in this methodological design as in all this research, it is important to distinguish between the design of the curriculum and the design of the ALMS, since the latter is only an instrument of the former.

- Finally, it should be noted that the objective of the design-based research method is the generation of theory from the reflection carried out in the research process.

CHAPTER 4. ANALYSIS OF RESULTS

This chapter presents the results obtained in this research together with an analysis of each one of them oriented toward the achievement of the objectives.

4.1 Test for characterization of modes of thinking

The learning condition chosen for the design of the curriculum was the modes of thinking in linear algebra (Sierpinska, 2000). To classify the students participating in the research according to this learning condition, a characterization test had to be designed[111] because, from a review of the existing psycho-technical tests it was possible to come to the conclusion that they use related concepts such as intelligence, aptitudes, problem solving or learning styles, factors that are not of central interest for the learning condition that was chosen.

Four factors were used to characterize each student's mode of thinking. The description of indicators used for each of the factors can be seen in Appendix 4.

Based on these indicators, items were constructed using the Likert scale methodology to measure the degree of agreement that each student had with two propositions presented in a dichotomous manner. Four items were constructed for each of the factors, which allowed not only classifying, but also characterizing the way of thinking of each student. As an example, one of the items is presented below.

[111] https://cutt.ly/EY4w4x3

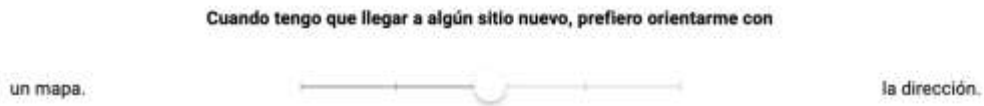


Figure 10. Example of a test item in the characterization of modes of thinking test.
Author's own elaboration (from the platform).

According to the test designed, in each of the items, the student must move the slider in the direction of the option with which he or she agrees the most. In this way, the student is classified in one of the two modes of thinking. As previously mentioned, Colombian students in the first semester of study have not developed the analytical-structural mode of thinking.

The validation of the test was carried out through two mechanisms: validation by experts and pilot testing. In the case of experts, a validation format was developed and applied with three people in the test construction field, to analyze the content validity of the test. The figure shows each of the criteria that the experts must evaluated for each item.

	0	1	2	3
Relevance. Refers to the factor to be evaluated.				
Relevance. Responds to the objective of the test.				
Relevance. Provides relevant information according to the objective.				
Adequacy. Easily understood (clear, precise, unambiguous, according to the level of information and language of the respondent).				
Adequacy. It is formulated for the recipients we are going to survey.				
Difficulty. Can be answered by a mathematics student.				
Bias. The response may be strongly oriented to social desirability.				

Figure 11. Criteria evaluated by the experts for each item.
Author's own elaboration.

The pilot test was applied to students of the same age range and knowledge as the students with whom the curriculum was subsequently implemented. The face validity of the test was analyzed with these students.

Once the test was applied to the students in each of the two iterations, the following classification results were obtained.

Table 7. Distribution of students according to modes of thinking.
Author's own elaboration.

Iteration	Synthetic-Geometric	Analytical-Arithmetic
1	47.6%	52.4%
2	40.6%	59.4%

From the table above it can be seen that the distribution among the modes of thinking is relatively homogeneous, which is surprising because one would think that due to the form of education received in Colombia, students should be more strongly inclined towards the analytical-arithmetic mode of thinking.

4.2 Comparative results of perception surveys

As mentioned above, to obtain information after each of the iterations, a perception survey was administered to students[112] and teachers. The survey measured the following for each of the stakeholders.

a. For students

   i. Platform: ease of navigation and aesthetics.

   ii. Methodology: clarity of teaching, quality of system moments and adaptive features.

   iii. Temporal factors: duration of the module and intensity of the work.

   iv. Perception of learning: sensation when learning autonomously and adaptively.

[112] https://cutt.ly/IY4w0Ul

b. For teachers

   i. Platform: ease of navigation, aesthetics, and functionality.

   ii. Methodology: adaptive features.

   iii. Temporal factors: program duration and intensity of work.

The complete framework developed for the design of the test for classifying students into the different modes of thinking can be found in Appendix 4.

The survey was applied to the teachers only in the first iteration since the teacher in the second iteration was the same researcher himself. In addition to the survey, an interview was conducted with them to determine the quality of the activities proposed in the nodes.

The data obtained from the application of the student survey after each iteration are shown below.

Table 8. Comparative results of the perception survey applied to students between iterations (data in percentages). Author's own elaboration.

	Iteration 1 (%)	Iteration 2 (%)
1. The way the learning moments are presented on the platform.		
It facilitated the reading of the learning contents.	13.9	9.5
It was complicated at times.	61.1	61.9
It was difficult to understand.	25.0	28.6
2. Navigation on the platform		
It allowed easy access to information.	47.2	66.7
At times it made it difficult to access information.	38.9	20.6
It was confusing.	13.9	12.7
3. The information presented		
It was clear in general.	16.7	19.0
It required being complemented by the teacher's explanation.	50.0	44.4
It was difficult to understand even with explanation.	33.3	36.5
4. In general, how did you find the level of difficulty of the comprehension questions and exercises?		
Easy	8.3	3.2
Adequate	33.3	46.0
Difficult	44.4	38.1
Very difficult	13.9	11.1
5. In general, how did you find the level of difficulty of the problems?		

Easy	0.0	0.0
Adequate	19.4	20.6
Difficult	66.7	63.5
Very difficult	13.9	15.9
6. Regarding your motivation during the learning process		
It always remained high, no matter what content or moments I was in.	5.6	9.5
It increased as I progressed and understood the dynamics better.	8.3	34.9
It declined as progress was made and there was an increase in difficulty.	66.7	42.9
It was low during the process.	19.4	12.7
7. Regarding the interaction with the teacher in the process of solving the problems you consider that		
It was required in the feedback of problems.	55.6	57.1
Other than on problems, it was required on more occasions.	33.3	28.6
It made no difference.	11.1	14.3
8. Regarding the interaction with your classmates during problem solving, do you consider that		
Enriched the consolidation of learning.	38.9	50.8
Sometimes obstructed or delayed the learning process.	36.1	38.1
Not necessary and could be a completely individual process.	25.0	11.1
9. The time given for learning in the subject module was		
Sufficient to achieve the personal learning expected by you.	44.4	42.9
Insufficient to achieve your personal learning expectations.	55.6	57.1
10. The amount of time you invested in the development of the module was		
higher than what I am used to in this area.	47.2	74.6
similar to what I am used to in this subject.	36.1	20.6
lower than what I am used to in this matter.	16.7	4.8
11. This way of learning seemed to you		
Not as good as the traditional way.	77.8	11.1
As good as the traditional way.	16.7	23.8
Better than the traditional way.	5.6	65.1
12. During your learning process, did you feel identified with the mode of thinking assigned to you by the adaptive system?		
All the time	2.8	7.9
Most of the time.	52.8	28.6
Some of the time.	27.8	49.2
A few times	11.1	9.5
Never	5.6	4.8
13. Classification in the mode of thinking		
It facilitated my learning process.	41.7	38.1
It was indifferent to my learning process.	41.7	42.9
It hindered my learning process.	16.7	19.0

Regarding question 1, in the first iteration it can be observed that a high percentage of students think that the presentation of the learning content was difficult to understand at least in some moments, this may be because students have poor reading skills, especially in reading mathematical texts. To address this weakness of the curriculum in the second iteration, video explanations were added to the text explanations, which however did not cause a difference in perception (although it should be noted that the groups in the two iterations had different students).

In the two iterations, most students consider the navigation of the platform and the clarity of the information presented to be acceptable.

The exercises proposed in the designed curriculum are of a routine nature, with the objective of familiarizing the student with the algorithms presented in the lessons. In this sense, it is surprising that a high percentage of students in both iterations stated that they found these exercises difficult.

The fact that a large proportion of the students indicated that motivation was low implies that the challenge problems probably did not generate commitment in them. However, this could also be due to the way the lessons were presented (in writing) and the fact that they had to work autonomously, without the teacher directly guiding the process. In any case, it can be seen that in the second iteration there was an increase in the perception of motivation, and although it is not clear what the causal factor is, it could be speculated that the fact that the students worked in groups helps them to stay motivated, given what is seen in question 8 in which report that working in groups enriched the process was significantly higher in the second iteration than in the first.

Many of the students value as necessary the interaction with the teacher for feedback on the problems and a good part of the students refer to it as necessary at other times as well. For this reason, in the second iteration, the possibility of students having contact with the teacher through a WhatsApp chat was increased. However, this did not cause a major difference in perception. It is risky to make speculations in this case, because just as there was a change of students, there was also a change of teacher between the two iterations.

In the first iteration a little more than half of the students responded that the time was insufficient to carry out the activities proposed in the module taught with this curriculum. This response may be biased by the fact that students dedicate less time than that established in the Colombian credit system for the independent study of the subjects, especially because a good part of them work while studying. Although in the second iteration the number of problems proposed to students in each node was rationalized (decreased), the response hardly varied (and even experienced a negative variation). It is worth noting that in both iterations more than half of the students did not manage to complete the entire process as designed; in general, the students only advanced as far as completion of the third node.

The response of the students participating in the first iteration to question 11 on the comparison between this way of learning and the traditional way of learning is very interesting. At the time the question was asked, the researcher intended the students to compare adaptive learning and learning according to the one-size-fits-all model, however, it is likely that other factors may have had a greater impact on the response than initially thought. Factors such as: interaction with a system in which the learning content is presented in written form; the absence of a teacher with the role of explaining the content when it is first encountered, where "explaining" means presenting examples that they could then replicate; having to solve non-routine problems, which is not a usual methodology in mathematics teaching at school or university level; or having to work in groups on some problems, which made it difficult for some students to advance at a fast pace. These factors are inherent to this curriculum design because it not only contemplates designing an adaptive platform, but also a curriculum that is applicable in adaptive environments. Such reactions to active methodologies are documented in the literature, e.g. in the article *Measuring actual learning versus feeling* *of learning in response to being actively engaged in the classroom*[113], in which it is shown that despite the fact that active learning is recognized as a superior method of classroom instruction, it has been detected that most STEM university professors still choose traditional teaching methods, even when it has been proven that when working within the first modality students learn more but feel that they learn less due to the greater cognitive effort required by active learning. However, in the second iteration some modifications were made to mitigate the influence of the factors described above: the difficulty of the non-routine problems presented to the students was reduced and working groups were formed from the beginning, which is reflected dramatically in their perception of the experience.

Based on how it was answered, another important question is number 12 in which it can be observed that the students participating in the first iteration felt identified, for the most part, with the mode of thinking in which they were classified by the characterization test. This contrasts with the answers to question 13 in that same iteration in which less than half of the students indicate that having been placed according to their thinking mode was convenient for the learning process, although it could also be considered that the students who answered that it was indifferent for their process did not note that it was detrimental to them, so they could be added to the previous percentage. In the second iteration the number of students choosing the first three options is very similar to that of the first iteration, although its distribution is different, since more students in the second iteration think that the grading system was only relevant sometimes, which is still paradoxical

[113] Deslauriers, L., McCarty, L. S., Miller, K., Callaghan, K., & Kestin, G. (2019). Measuring actual learning versus feeling of learning in response to being actively engaged in the classroom. Proceedings of the National Academy of Sciences, 116(39), 19251-19257.

since it does not correspond with the answer to the last question in which a high percentage (38.1%) points out that the curriculum facilitated their learning process.

For their part, the teachers referred that the problems were difficult for the average level of students and that there were too many problems for them to solve in the expected time, made some suggestions on the presentation of the contents, which were welcomed by the researcher, and valued the classification of the students into modes of thinking.

4.3 Analysis of the solution of the challenging problems

The core of the designed curriculum, in terms of the development of mathematical thinking, is the solution of challenging problems.

In general, in each node corresponding to each thinking mode the challenging problems were proposed after the students had had the opportunity to read the learning content and/or watch the related videos. Only in some cases were the problems used as a pretext to introduce content. For example, in node 4, which is framed in the analytical-structural mode of thinking for all students, the introduction of the concept of vector space is made after the students have had the experience of solving the problem presented in the following figure.

Problema 1. *A condition for a set to be a vector space is that the operations of addition and product by a scalar are closed on that set.*

Consider the set $\mathbb{R}^2$ with the usual operations of vector addition and product of a scalar ($\mathbb{R}$) by a vector. Give an example of a subset of $\mathbb{R}^2$ that complies with:

1. *is closed under addition (i.e. the sum of two vectors in this subset of $\mathbb{R}^2$ results in a vector in the subset), but not under multiplication by a scalar.*
2. *be closed under multiplication by a scalar (that is, multiplication of a vector in the subset of $\mathbb{R}^2$ by a scalar results in a vector in the subset), but not under vector addition.*

Graphically represent these sets.

Figure 12. Problem 1 of node 4.

Author's own elaboration.

On the other hand, although a student (or group of students, in the case of the second iteration) are classified in a particular mode of thinking, this did not imply that the content and problems presented to him or her were exclusively framed in that mode of thinking, but that the first approach to the content and the first problems posed corresponded to that mode, given that the intention was that the student make changes of representation between the different modes, because this is a characteristic manifestation of mathematical thinking. That is, a student who was classified in the geometric mode of thinking was initially presented with the learning content in that mode, but it was gradually proposed that he or she make representational changes, not only in the learning content, but also in the challenging problems.

In addition to this first characteristic manifestation of mathematical thinking, four other characteristics were detected in the review of the solutions that students proposed to the problems.

Modeling problems in mathematical objects was one of them. When confronted with problems, most students were able to identify the mathematical object that modeled the situation. This is an initial step in the application of mathematical thinking, which does not necessarily imply that the problems are correctly solved. In this case, the following figure shows an example of a group of students who did not correctly model a problem.

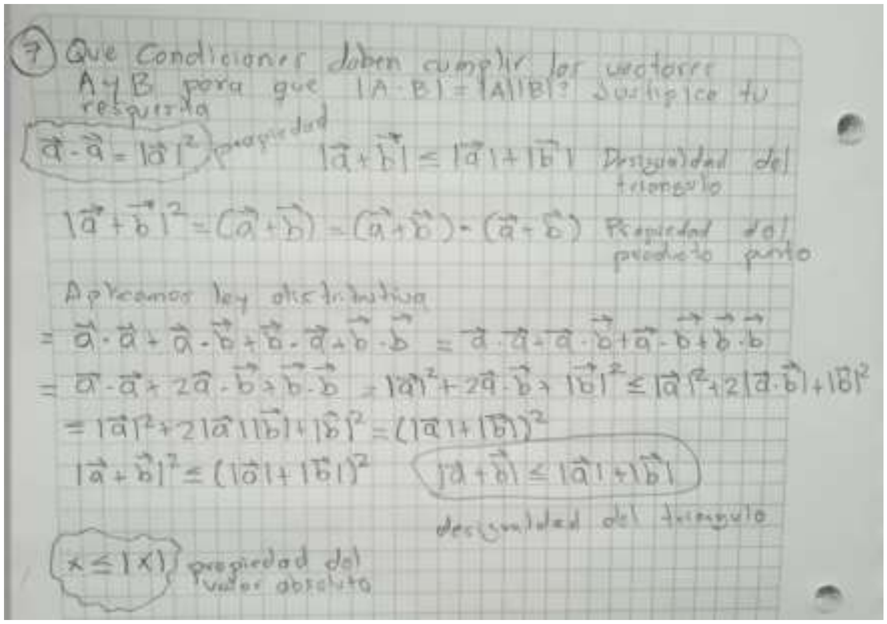


Figure 13. Solution of problem 7 of node 3 of the arithmetic mode of thinking.

Author's own elaboration.

In this solution the students were supposed to model the problem using the geometric presentation of the dot product, however they make use of the triangular inequality and the property of the dot product of a vector with itself.

In other cases, the students made use of previous learning from other subjects to model some of the proposed problems. That is not a mistake but would be more desirable had they made use of the learning content that was at stake in this module. (On the other hand, this shows recursiveness in the students.) An example of this can be seen in the following figure in which the students do not solve the problem correctly, however, they model it well using mathematical objects of differential calculus (optimization of functions).

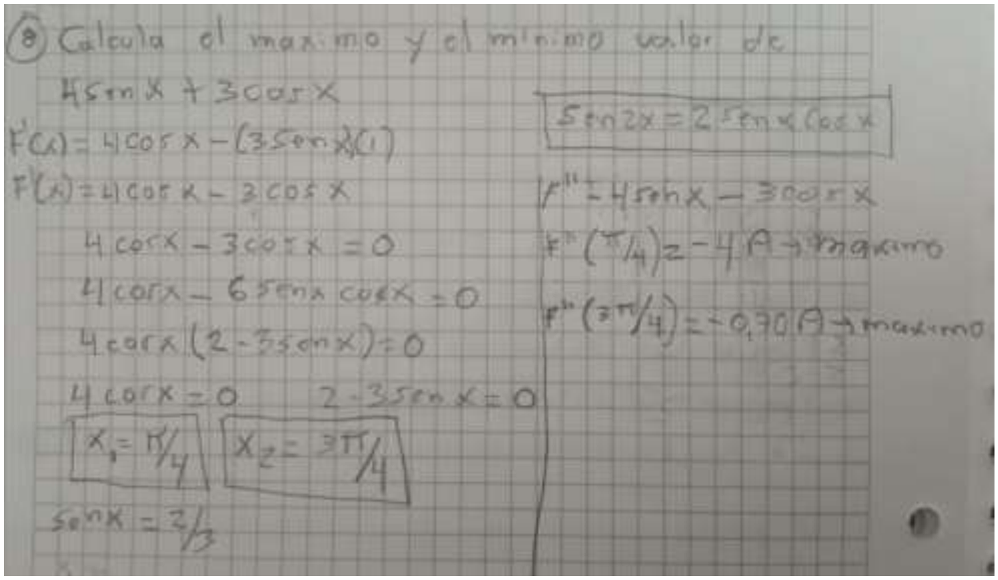


Figure 14. Solution of problem 8 of node 3 arithmetic.

Author's own elaboration.

Mathematical thinking is also evident in the correct formulation of strategies to solve problems. This is something that is difficult for students to do when faced with challenging problems (when the strategy is not obvious), because they are highly accustomed to solving routine problems for which they know the strategy. For example, in the following problem students correctly model the problem, but do not use a good strategy. In general, in this problem, students searched for vectors with coordinates in the integer grid, which resulted in a correct solution of the problem, but not with a correct strategy.

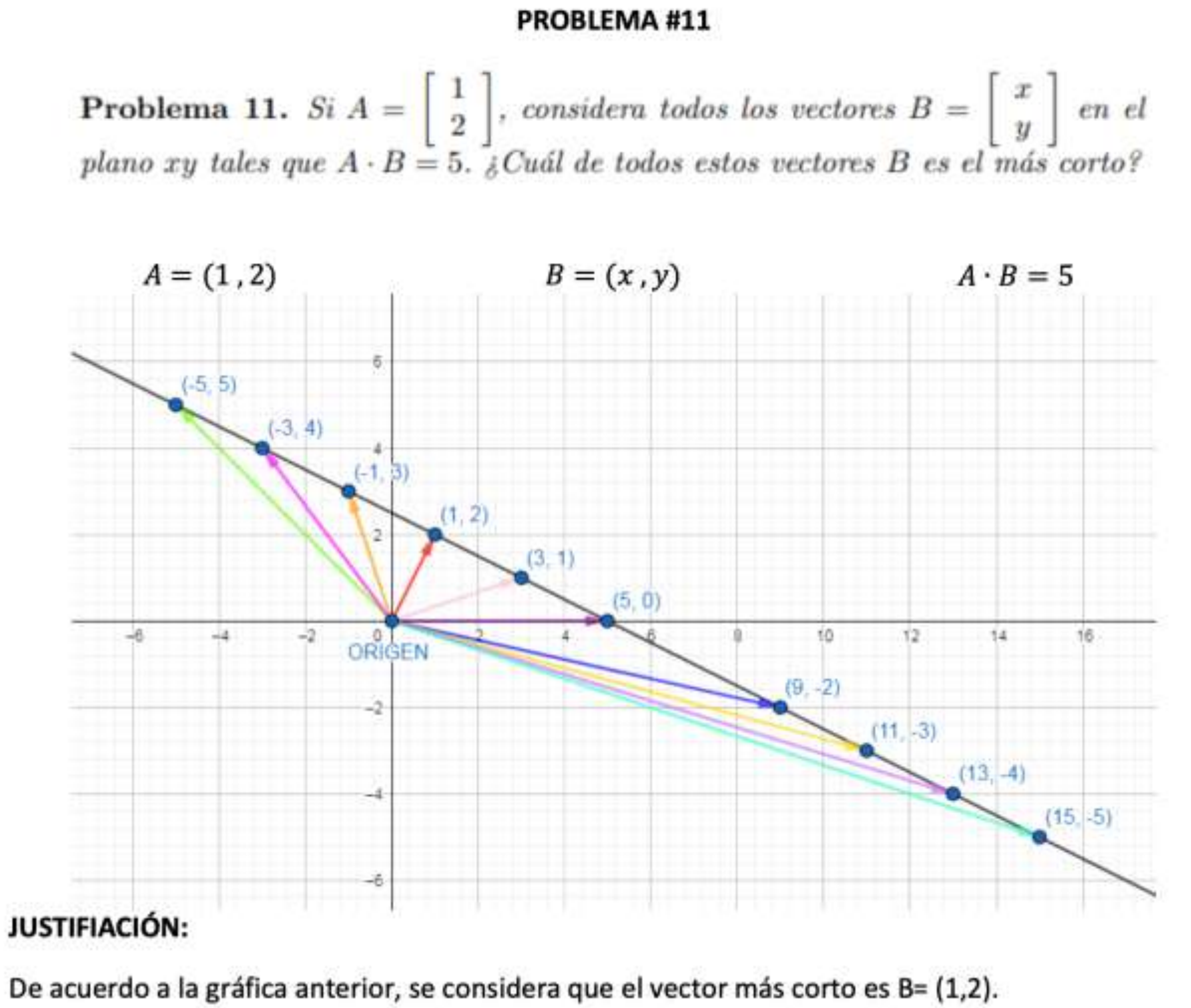


Figure 15. Solution of problem 11 of node 2 arithmetic mode.

Author's own elaboration.

Another example, this time of formulating a good strategy to solve a problem, can be seen in the following figure. In this solution the students correctly model the problem and solve it using a relevant strategy which is not obvious.

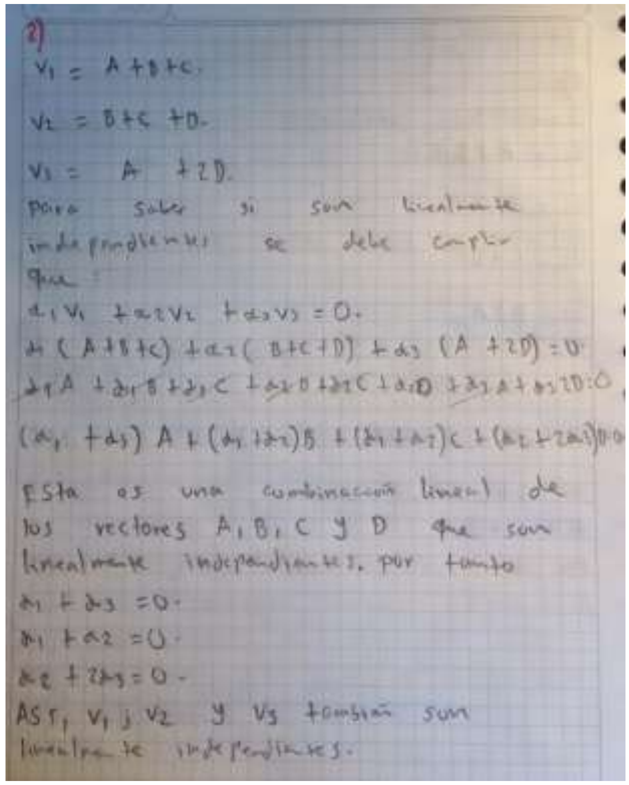

2)
$V_1 = A + B + C.$
$V_2 = B + C + D.$
$V_3 = A + 2D.$
Para saber si son linealmente independientes se debe cumplir que:
$\alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3 = 0.$
$\alpha_1 (A + B + C) + \alpha_2 (B + C + D) + \alpha_3 (A + 2D) = 0$
$\alpha_1 A + \alpha_1 B + \alpha_1 C + \alpha_2 B + \alpha_2 C + \alpha_2 D + \alpha_3 A + \alpha_3 2D = 0$
$(\alpha_1 + \alpha_3) A + (\alpha_1 + \alpha_2) B + (\alpha_1 + \alpha_2) C + (\alpha_2 + 2\alpha_3) D = 0$
Esta es una combinación lineal de los vectores A, B, C y D que son linealmente independientes, por tanto
$\alpha_1 + \alpha_3 = 0.$
$\alpha_1 + \alpha_2 = 0.$
$\alpha_2 + 2\alpha_3 = 0.$
Así, V_1, V_2 y V_3 también son linealmente independientes.

Figure 16. Solution of problem 3 of node 3 arithmetic mode.

Author’s own elaboration.

Correct communication of the solution of mathematical problems is a characteristic with which mathematical thinking can be evidenced; at first glance this would seem not to be so relevant, but once a pattern of regularity in a student's performance with respect to this factor is observed, it can also be noticed that their mathematical thinking has developed at a good level. Reciprocally, in some cases students solve the problems, but the communication of the solution is poor, from which it can be inferred that, although they know the methods, they do not have a clear idea of the purpose of what they are doing.

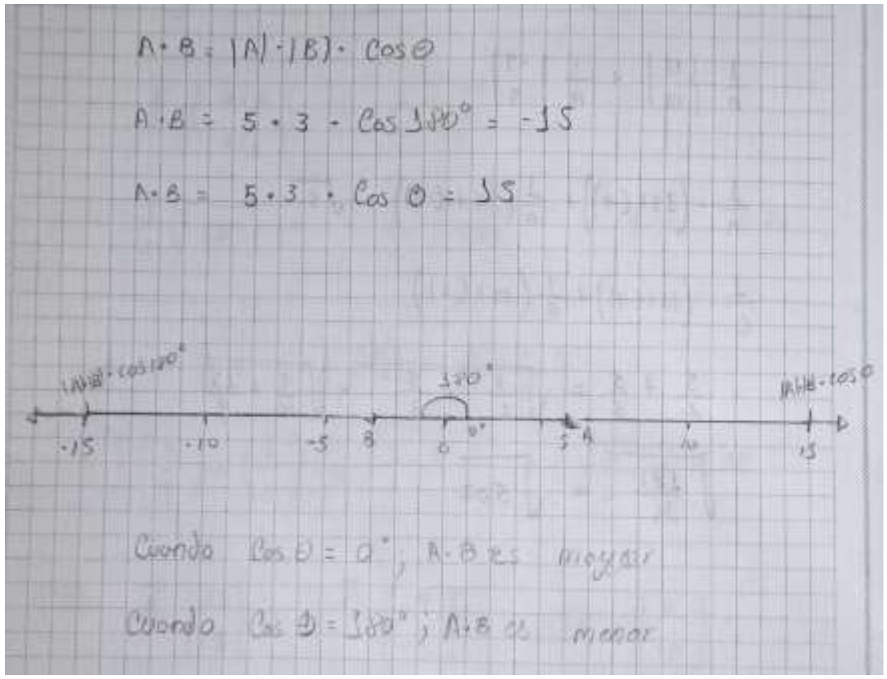


Figure 17. Solution of problem 1 of node 2 geometric.

Author's own elaboration.

In the previous problem, the students had to find the largest and smallest value of the expression in the first line, considering that $|A| = 5$ y $|B| = 3$. As can be noticed, already in the third line the problem was solved, however, they make a change of representation, very much in accordance with their mode of thinking, and communicate the answer in a more complete way by making an arithmetic analysis of the geometric form presented.

Finally, and in line with the APOS theory, it is possible to note different levels in the structuring of mathematical thinking; one level in which actions, which are constructed when repeatedly given responses to stimuli, evolve into processes through the internalization of actions; another level in which objects are constructed by encapsulating processes; and a level in which actions, processes and objects are abstracted to give rise to schemas.

For example, in problem 8 of node 1 of the geometric mode of thinking, students are asked to find the norm of the vectors |A|B and |B|A based on two particular vectors that they propose. Then, in order to have this action evolve into a general process, it is proposed to them that they do the same but with generic vectors. And, finally, they are asked to give a conclusion based on what they have done, with the objective of encapsulating it into an object. In the following figure (the complete resolution of the problem is not shown) it can be seen that the students manage to put the action into practice and evolve it into a process. However, in the moment of encapsulating the object, they talk about the commutative property, which is only conceived for operations and therefore is not pertinent in this case. Nevertheless, it is a first conception of the object.

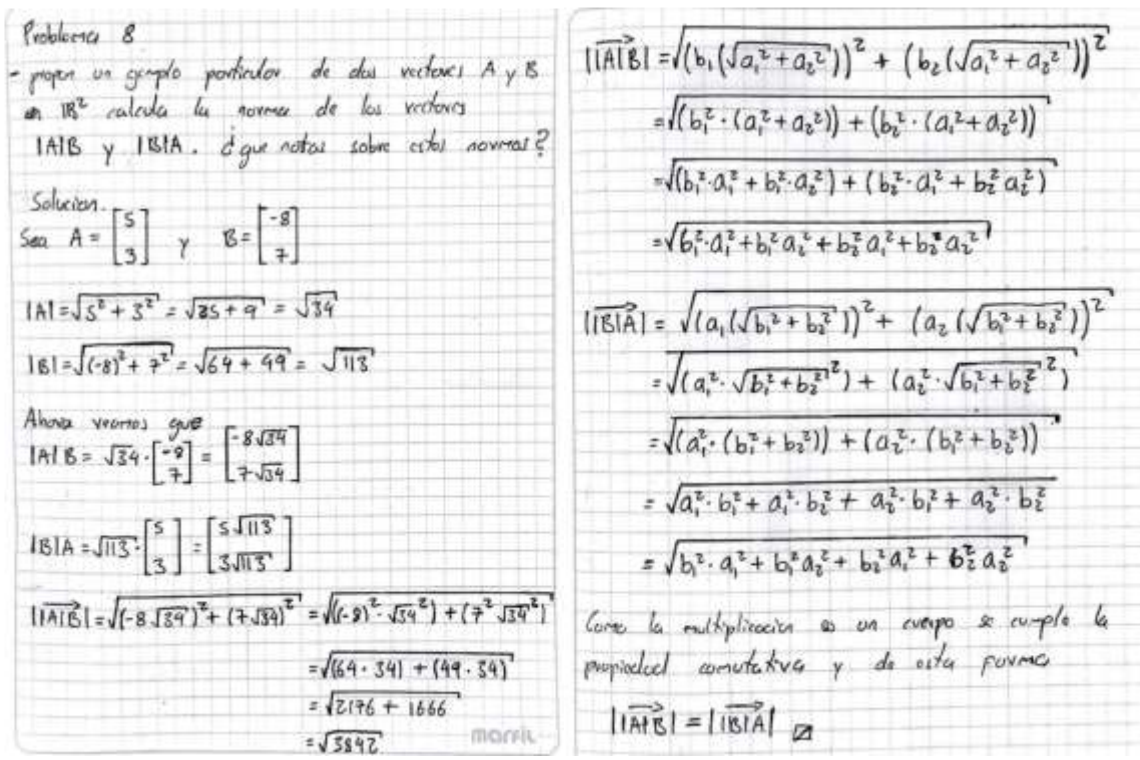


Figure 18. Solution of problem 8 of node 1 geometric mode.

Author's own elaboration.

Problem 8 of node 1 of the arithmetic mode of thinking was proposed in order to evidence the use of schemes by students, in this case the scheme generated by the dot product. In the following image the statement of the problem can be seen.

Problema 8. *In physics, the law of levers is expressed as follows:* $P \times B_P = R \times B_R$*, where* P *means the force applied (power),* R *means the force or weight to be moved (resistance),* B_P *is the distance from the fulcrum to the point of application of the power and* B_R *is the distance from the fulcrum to the point of location of the resistance, as can be seen in the following image.*

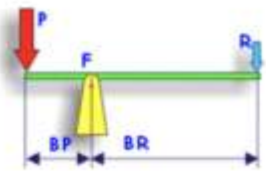


Figura 9: Elements of a lever

Use the dot product to model this law.

Figure 19. Problem 8 of node 1 arithmetic mode.

Author's own elaboration.

This problem was particularly difficult for most of the students, since having a scheme implies recognizing its parts and being able to use them properly, i.e., it is required to recognize which objects are involved and which processes with those objects should be used.

The characteristics of the manifestation of mathematical thinking that have been presented in the previous examples can be condensed into a rubric, which rather than serving to qualify the solution of individual problems, serves to characterize the forms and levels of mathematical thinking of a person based on a large enough number of problems solved. This rubric is an original proposal of the present research and can be seen in the following image.

Rubric for observation of the development of mathematical thinking.

Dimension	Low level	Medium level	High level
Changes in the representation of mathematical objects	It does not establish relationships between ways of thinking	Establishes relationships between arithmetic and geometric modes of thinking	Establishes relationships between arithmetic, geometric and structural modes of thinking.
Modeling of problems in mathematical objects	Does not demonstrate knowledge of the characteristics of mathematical objects.	Demonstrates knowledge of the characteristics of mathematical objects but does not use them effectively to model problems	Models problems using pertinent mathematical objects
Formulation of problem-solving strategies	Does not propose or proposes loose ideas about problem-solving strategy	Proposes complete strategies to solve problems	Proposes insightful or diverse strategies for solving problems
Development of problem-solving strategies	Have difficulties following formulated problem-solving strategies	Makes minimal errors in following proposed problem-solving strategies	Fully develops problem-solving strategies
Communication of mathematical problem solving	Have difficulties in explaining problem solving	Clearly explains problem solving	Argues for problem solving
Thought structuring	The evolution of actions in processes is evidenced	The encapsulation of processes in objects is evidenced.	The generation of thinking schemes is evidenced

Figure 20. Rubric for observation of the development of mathematical thinking.

Author's own elaboration.

This rubric is an essential element in the design of an adaptive curriculum that aims not only to measure performance, but also to develop mathematical thinking. The rubric not only allows us to characterize the mathematical thinking of the participants, but also acts as a guide for the design of the instruments that lead to the development of mathematical thinking.

4.4 Characterization of the curriculum designed based on Newman's taxonomy

The taxonomy proposed by Newman (2013) allows us to characterize the adaptive curriculum proposed in this research, as follows.:

- Learner Profile (or student model): in this case, the system only obtains information about the student's profile at the beginning of the learning process, but it is not updated during the learning process.
- Unit of Adaptivity: although this curriculum was designed and implemented to develop a single subject module, it is possible to extrapolate this to adapt the learning content of the full course.

- Instruction Coverage: the curriculum designed can cover the teachings of the complete linear algebra course, and can even be extrapolated, with the respective changes in learning conditions, to other courses of a program of study.
- Assessment: due to technical limitations, this curriculum does not allow for continuous or adaptive evaluation, however, it does reach the level of formative evaluation.
- Content Model (or domain model): the adaptable system designed is closed in its domain model, that is, it is not the teacher who modifies the contents presented in the system, although it has certain flexibility in its modification.
- Bloom's Coverage: although there are other taxonomies of thought processes, following Newman one could say that this curriculum is at the creation/evaluation level, i.e., the highest level in Bloom's taxonomy.

4.5 On the design of adaptive curricula that fosters the development of mathematical thinking, considering the learning conditions of the students

The last phase of the design-based research method is documentation and reflection to generate design principles.

4.5.1 A definition of adaptive curriculum

Before proposing a methodology for the design of adaptive curricula that foster the development of mathematical thinking considering the learning conditions of the students, it is necessary to agree on a definition of adaptive curriculum that outlines the design principles that emerged from the research. Obviously, this definition must be derived from the definition of curriculum, however there are many definitions of curriculum (for sample: https://cutt.ly/gY5Ghfv).

In the present research, the curriculum is considered to be the set of elements that allow the development of the educational process toward a certain end. This is a broad definition that includes, for example: the purposes of the process, the way to evaluate the achievement of these purposes, the learning content, sequences, methodologies, and didactic resources, that constitute the domain model.

In this sense, an adaptive curriculum includes, in addition to the above elements, the model of the student for whom the curriculum is to be adapted (in which the characteristics of the students and the learning conditions chosen to make the adaptation are specified) and the characteristics of the adaptation model in the terms that Newman (2013) proposes.

There is a difference between the learning management system, whether adaptable or adaptive, and the curriculum. The former is a resource in which the elements of the latter are articulated; however, a curriculum is much more than just the repository that contains it.

4.5.2 Emerging design principles

From the design of the curriculum for the linear algebra module and its implementation in the two iterations, these design principles emerge:

- Before thinking about the design of the LMS, whether adaptable or adaptive, it is necessary to be clear about the elements of the curriculum (in the sense of the definition previously presented).
- In the current conditions of technological development, the design of an adaptive curriculum is an interdisciplinary task shared by professionals in the teaching/learning of the specific contents and professionals in the use of information and communication technologies, at least.

- An adaptive curriculum that aims to develop mathematical thinking must have a way to characterize this thinking in students. This implies that if the curriculum is deployed in an adaptive LMS, i.e., one whose adaptive processes are dynamic in accordance with student progress, progress decisions must be made based on this characterization and not only from the performance results.
- An adaptive curriculum oriented to the development of mathematical thinking should consider solving challenging problems. These problems should be designed according to the evident characteristics of mathematical thinking.
- An important element to design in adaptive curricula is the role of the teacher, since in this type of curricula the role of the teacher evolves but is not marginalized.
- An adaptive curriculum does not have to involve individual learning. The benefits of communication among students and between students and the teacher should be taken advantage of in the design of this type of curriculum. Moreover, due to the characterization of the students' learning conditions, it is possible to establish more fruitful relationships if interactions between people in the same group and interactions between people classified in different groups are planned.

4.5.3 A methodology for the design of adaptive curricula that fosters the development of mathematical thinking, considering the learning conditions of the students

Therefore, based on the results obtained and their respective analysis and on the design principles proposed in the preceding section, a methodology for the design of adaptive curricula that foster the development of mathematical thinking, considering the learning conditions of the students, is proposed as follows. It is important to note that the procedure to be described cannot be exhaustive, therefore it does not pretend to be so, due to the number of variables that arise in a real educational process.

1. For the design of the student model:
    a. Determine the characteristics of the students in terms of, at least, level of knowledge and learning interests. For this purpose, a characterization test can be designed and implemented.
    b. Determine the learning conditions on which the educational process will be adapted. Since the curriculum aims at the development of mathematical thinking, the learning conditions must be related to this factor.
2. For the design of the domain model:
    a. The central purpose of this kind of curriculum will be the development of mathematical thinking; however, there may be complementary purposes. If there are, determine each such complementary purpose of learning.
    b. Determine whether the students will work individually or in groups. If groups are to be formed, determine whether they will be made up of people from the same classes emerging from the learning conditions or whether they will be groups with people from mixed classes.
    c. Determine the characteristics of evaluation, since it can be continuous and in articulation with the adaptation, or it can be formative throughout the process, but without influencing the way in which adaptation has been implemented. It is also necessary to determine, based on the above, the type of instruments to be used for the evaluation. It is important to keep in mind that assessment can be differentiated according to the classes emerging from the learning conditions or it can be universal.

d. Determine the framework of content to be taught. This can be considered between lessons or course levels; even more global levels can be taken into account.

e. Perform a genetic decomposition of the framework of learning contents that are going to serve as a medium for the development of mathematical thinking.

f. Articulate the result of the genetic decomposition and the characteristics of the chosen learning conditions, considering that the nature of these learning conditions may vary the order proposed by the genetic decomposition.

g. Determine whether the content model is going to have authoring tools or is going to be a closed model, to be elaborated *a priori*.

h. Design the ways to present (deductively) or stimulate the construction of the learning content (inductively) according to the classes emerging from the learning conditions.

i. To design the challenging problems, consider the rubric for the observation of the development of mathematical thinking, the characteristics of this type of problems and the decision of the previous point. It is important to note that the same contexts can be used for several problems that are located in different classes from those formed by the learning conditions that were taken into account for the adaptation, making the way in which the solution is asked to be found different for each one.

3. For the design of the adaptation model:

   a. Determine the role of the teacher and the moments of interaction with the student.

   b. Classify students according to the chosen framework of learning conditions. Sometimes this classification will involve the design and implementation of instruments to determine characterization.

c. Determine whether the model is to be static or dynamic. If it is decided that it will be dynamic, determine the degree of adaptation, e.g., determine whether adaptation will occur after a certain group of lessons, or after each lesson, or even within each lesson.

4. For the design of the LMS:

   a. Determine whether to adapt an LMS or to create one.

   b. Determine how communication will be between students and between the teacher and students in the LMS. Decide as to whether the process will be of an individual or group nature. In the case where individual work is preferred, group activities can always be designed, for which it is necessary to determine whether these activities will be carried out among students of the same emerging class based on the learning conditions or whether these classes can be mixed.

An analogy can be made between this methodology and a gear mechanism. Such a mechanism would be composed of three sets of gears: the set of the student model, the set of the domain model and the set of the adaptation model, arranged on a support that gives shape to the whole machine, which in this case is the LMS. And continuing with the analogy, the machine has an already established form, however, there are certain gears that are interchangeable and that generate options in the production of the machine. For example, in the set of gears that make up the student model, one can change the type of learning conditions that will be considered to make the adaptation. Similarly, in the set of gears that make up the domain model, one can change, among others, the contents, the methodology and the way in which the evaluation will be carried out. Finally, in the set of gears that make up the adaptation model, one can choose whether to use the static adaptation gear or the dynamic adaptation gear, to give an example. These gear changes allow (as in 'real' mechanisms) adapting speeds, intensities, or force distributions, which in this case would allow to 'go at the pace' of the learner. This analogy makes it possible to see that the system is quite flexible, but that it maintains its structure oriented towards one purpose: to develop students' mathematical thinking while considering their learning conditions.

As final notes on this adaptive curriculum design methodology for the development of thinking, it is important to document the whole process in a complete and orderly manner and to have a flexible attitude because during the design process, and even during the implementation process, conditions could change, which would lead to having to make modifications to the decisions taken.

Chapter conclusions

Based on the analysis of the research results, this chapter presents a methodology for the design of adaptive curricula oriented to the development of mathematical thinking.

CONCLUSIONS

Based on the design-based research method, this paper proposes a methodology for the design of adaptive curricula that foster the development of mathematical thinking considering the students' learning conditions. This methodology is quite flexible in its conception, in the sense that it allows making decisions in the characterization of the students in the adaptive factors to be taken into account, in the way of structuring the learning contents and in other associated pedagogical factors, however, it establishes a prescriptive framework for the construction of this type of curricula.

The research explored various learning conditions from which it is possible to adapt a curriculum, and the decision was made to use the modes of thinking proposed by Sierpinska (2000), since they are relevant to the fundamental purpose of the proposed curriculum, which is the development of mathematical thinking, and are suitable for modeling the learning of the contents of linear algebra. The modes of thinking do not depend only on the cognitive forms of the students, but also on the epistemological form of the learning contents.

To classify students in the modes of thinking, a test was designed to determine whether a student is more inclined towards the synthetic-geometric mode of thinking or towards the analytical-arithmetic mode. The application of this test showed that, contrary to what could be hypothesized (that there would be a majority inclination towards the analytical-arithmetic mode of thinking), students are quite homogeneously distributed between these two categories.

A fundamental characteristic of adaptive curricula is the need for articulation between the learner model, the domain model, and the adaptive model. Regardless of the decisions made in terms of the adaptive factors and the way in which the learning content is structured, we must not lose sight of the fact that the objective of the curriculum is to foster the development of students' mathematical thinking.

To structure the learning contents, the theoretical framework related to the genetic decomposition of the APOS model of Dubinsky (1991) was used. This model was used taking into account not only the epistemological characteristics of the learning contents, but also the characteristics of the students' modes of thinking.

The proposed methodology integrates important novelties with respect to what is known in the state of the art. These novelties are:

- Unlike the adaptive curricula reviewed, which base their adaptation and outcomes on student performance, the curriculum proposed in this research proposes a framework for the characterization of students' mathematical thinking and uses adaptive factors associated with this.
- An active role for the teacher is proposed, i.e., while several of the adaptive systems reviewed seek automation and, therefore, the exclusion of the teacher, the system proposed in this research assigns a role to the teacher in the teaching-learning process.
- The adaptive systems reviewed privilege individual work, however, the curriculum proposed in this research recognizes the importance of communication among students for the development of their mathematical thinking.
- The adaptive curriculum proposed in this thesis uses as a resource for the development of mathematical thinking the resolution of challenging problems. In addition, the document that records the research process presents the result of an inductive analysis of the characteristics of this type of problems.

From the way the learning contents were sequenced and from the two classes into which the students were divided (synthetic-geometric and analytical-arithmetic), one might think that the mode of thinking is influenced by the level of knowledge, but this only occurs when knowledge has been achieved by traditional methods, which initially privilege the graphic and then the arithmetic. However, there are cultures in which the approaches have been carried out in different ways, which would prove that the mode of thinking is not necessarily influenced by the level of knowledge.

In this research, a rubric was proposed to detect students' mathematical thinking, particularly when solving challenging problems. Based on what was reported in this thesis, it can be seen that students put into play, at different levels, their mathematical thinking when solving these problems.

A group of conditions that were not considered at the beginning of the design of the curriculum in this research were imposed by the prevailing educational paradigms of Colombian students. Paradigms that have had consequently, among students, the lack of level in reading comprehension of general and mathematical texts, the lack of autonomy in independent work and negative attitudes when facing problems that imply non-routine actions. These conditions negatively affected the way students perceived the process of adaptive education.

The context of the proposed methodology for the design of adaptive curricula is limited to mathematics subjects, but it could be extrapolated to subjects in other disciplinary fields, given the importance and relevance of adaptive education.

Finally, it is important to note that this research was conducted under conditions of the pandemic caused by the SARS-CoV-2 virus, which did not represent major changes for its development, since adaptive education through intermediate platforms can be carried out both in person and remotely.

RECOMMENDATIONS

Based on the research experience, the following recommendations are proposed for future research in this area:

- About ALMS improvements:
    - Implement the possibility for students to send the solution of the problems through the ALMS and not through external means.
    - Implement the possibility of giving written feedback to students, so that they have supporting documents that can guide them in the learning process.
    - Explore the adaptation of the learning paths using technologies such as machine learning.
- Although this curriculum was designed for content mastery at the university level, designs could be explored for other ages where, possibly, there is less inertia in factors such as difficulty in reading comprehension (and mathematical reading) and autonomy in the learning process.
- The possibility of measuring the effectiveness of an adaptive curriculum created with this methodology remains open.
- Regarding learning conditions, it could be interesting to carry out studies on the correlation between different classifications, for example, modes of thinking and gender, modes of thinking and age, etc.
- In the case where an adaptive curriculum is to be implemented in several generations of students, care must be taken with the renewal of the didactic material, since there is a risk of transmission of the material between generations.

REFERENCES

Aiken, L. (1996) *Test psicológicos y evaluación*. México: Prentice Hall Hispanoamericana.

Alameen, A., & Dhupia, B. (2019). Implementing Adaptive e-Learning Conceptual Model: A Survey and Comparison with Open Source LMS. *International Journal of Emerging Technologies in Learning (iJET), 14*(21), 28-45.

Alshammari, M. T., & Qtaish, A. (2019). Effective Adaptive E-Learning Systems According to Learning Style and Knowledge Level. *Journal of Information Technology Education*(18), 529-547.

Alzain, A. et al. (2018). Adaptive education based on learning styles: are learning style instruments precise enough? *International Journal of Emerging Technologies in Learning,, 13*(9), 41-52.

Arnon, I. et al. (2014). *APOS theory. A Framework for Research and Curriculum Development in Mathematics Education.* New York: Springer.

Arroyo, I. et al. (2014). A multimedia adaptive tutoring system for mathematics that addresses cognition, metacognition and affect. *International Journal of Artificial Intelligence in Education, 24*(4), 387-426.

Artigue, M. (1989). *Épistémologie et didactique.*

Berman, A. (2018). Using Challenging Problems in Teaching Linear Algebra. In *Challenges and Strategies in Teaching Linear Algebra* (pp. 369-378). Cham: Springer.

Bertúa, J., & Denenberg, M. (2016). Un cambio metodológico y de contenidos en álgebra lineal. *Revista de Educación Andrés Bello* (4), 54-86.

Bettahi, J. (2018). Personalized Learning: One-Size-Fits-One Model.

Brown, A. L. (1992). Design experiments: Theoretical and methodological challenges in creating complex interventions in classroom settings. *Journal of the Learning Sciences, 2*(2), 141–178.

Brown, F. (1980) *Fundamentos de Medición y Evaluación en Psicología y Educación*. México: El Manual Moderno.

Brusilovsky, P. (2000). Adaptive hypermedia: From intelligent tutoring systems to Web-based education. In *International Conference on Intelligent Tutoring Systems* (pp. 1-7). Berlin, Heidelberg: Springer.

Brusilovsky, P. (2001). Adaptive Hypermedia. *User Modeling and User Adapted Interaction, 11*, 87-110.

Brusilovsky, P., & Maybury, M. (2002). From adaptive hypermedia to the adaptive web. *Communications of the ACM, 45*(5), 30-33.

Brusilovsky, P., & Peylo, C. (2003). Adaptive and intelligent web-based educational systems . *International Journal of Artificial Intelligence in Education, 13*(2-4), 159-172.

Cantoral, R. et al. (2005). *Desarrollo del pensamiento matemático.* México: Trillas.

Carlson, D. et al (1993). The Linear Algebra Curriculum Study Group recommendations for the first course in linear algebra. *The College Mathematics Journal, 24*(1), 41-46.

Chapman, O. (2011). Supporting the development of mathematical thinking. In B. Ubuz, *Proceedings of the 35th International Conference for the Psychology of Mathematics Education* (pp. 69-75). Ankara, Turkey: PME.

Cobb, P. et al. (2003). Design Experiments in Educational Research. *Educational Research, 1*(32), 9-13.

Coffield, F., Moseley, D., Hall, E., and Ecclestone, K. (2004b). Should We Be Using Learning Styles? What Research Has to Say to Practice. Learning and Skills Research Centre / University of Newcastle upon Tyne., London.

Corno, L. Y. (2008). On teaching adaptively. Educational Psychologist, 43(3), 161-173.

Datta, S., & Sengupta, S. (2018). A Review on the Adaptive Features of E-Learning. *International Journal of Learning and Teaching, 4(4)*, 277-284.

Dembo, M. H., & Howard, K. (2007). Advice about the use of learning styles: A major myth in education. *Journal of college reading and learning, 37*(2), 101-109.

Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher, 32*(1), 5-8.

Deslauriers, L., McCarty, L. S., Miller, K., Callaghan, K., & Kestin, G. (2019). Measuring actual learning versus feeling of learning in response to being actively engaged in the classroom. *Proceedings of the National Academy of Sciences,* 116(39), 19251-19257.

Devlin, K. (2012). *Introduction to mathematical thinking.* Petaluma, CA.

Dhakshinamoorthy, A., & Dhakshinamoorthy, K. (2019). KLSAS—An adaptive dynamic learning environment based on knowledge level and learning style. *Computer Applications in Engineering Education, 27*(2), 319-331.

Dohring, D. C., Hendry, D. A., Gunderia, S., Hughes, D., Owen, V. E., Jacobs, D. E., ... & Salak, W. (2019). U.S. Patent No. 10,490,092. Washington, DC: U.S. Patent and Trademark Office.

Donevska-Todorova, A. (2015). Conceptual Understanding of Dot Product of Vectors in a Dynamic Geometry Environment. Electronic Journal of Mathematics & Technology, 9(3).

Doroudi, S. (2019). *Integrating Human and Machine Intelligence for Enhanced Curriculum Design.* Doctoral dissertation, Air Force Research Laboratory.

Drijvers, P. (2015). Denken over wiskunde, onderwijs en ICT. [Thinking about mathematics, education and ICT.] Inaugural lecture. Utrecht, the Netherlands: Universiteit Utrecht.

Drijvers, P. et al. (2019). Assessing mathematical thinking as part of curriculum reform in the Netherlands. *Educational Studies in Mathematics*, 1-22.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking . In D. Tall, *Advanced Mathematical Thinking* (pp. 95-123). Dordrecht: Kluwer.

Dubinsky, E. (1997). Some thoughts on a first course in linear algebra at the college level. *MAA NOTES*, 85-106.

Durlach, P. J. (2019). Fundamentals, Flavors, and Foibles of Adaptive Instructional Systems. In *International Conference on Human-Computer Interaction* (pp. 76-95). Cham: Springer.

Duval, R. (2017). *Understanding the Mathematical Way of Thinking–The Registers of Semiotic Representations.* Springer International Publishing.

El-Bishouty, M. M. (2019). Use of Felder and Silverman learning style model for online course design. *Educational Technology Research and Development, 67*(1), 161-177.

El-Hadad, G. et al. (2019). Adaptive Learning Guidance System (ALGS). *arXiv preprint arXiv:1911.06812.*

Fatahi, S., & Moradian, S. (2018). An Empirical Study on the Impact of Using an Adaptive e-Learning Environment Based on Learner's Personality and Emotion. *International Association for Development of the Information Society.*

Felder, R. M., & Silverman, L. K. (1988). Learning and teaching styles in engineering education. *Engineering education, 78*(7), 674-681.

Ferrera, A. (2008). Estrategias de aprendizaje, construcción y validación de un cuestionario-escala. Universidad de Valencia. En https://www.tdx.cat/handle/10803/10306#page=1

FitzGerald, E. et al. (2018). A literature synthesis of personalised technology-enhanced learning: what works and why. *Research in Learning Technology, 26*, 1-16.

Fröschl, C. et al. (2008). Learner Model in Adaptive Learning. *World Academy of Science, Engineering and Technology* (21).

Gagné, R. M. (1975). *Essentials of learning for instruction.* Dryden Press.

García-Hurtado, O. et al. (2019). Linear algebra learning focused on plausible reasoning in engineering programs. *Visión electrónica, 13*(2), 1-22.

Gascueña, J., Fernández-Caballero, A., & González, P. (2005). *Ontologías del modelo del alumno y del modelo del dominio en sistemas de aprendizaje adaptativos y colaborativos*. Castilla, España.

Glaser, R. (1977). *Adaptive education: Individual diversity and learning*. New York: Holt, Rinehart & Winston.

Godino, J. D. (2013). La ingeniería didáctica como investigación basada en el diseño. *CERME.*

Gómez, S., & Roquet, J. V. (2012). *Metodología de la investigación.* México: Red tercer milenio.

Gonzalez, M. P. (2019). Personalización y adaptación en un ambiente virtual de aprendizaje basada en estilos, conocimiento previo y errores frecuentes. *XXI Workshop de Investigadores en Ciencias de la Computación*.

González, O. (1995). Curriculum: diseño, práctica y evaluación. *CEPES*.

Graf, S. (2007). *Adaptivity in learning management systems focussing on learning styles.* Vienna: Vienna University of Technology.

Graf, S. et al. (2009). Investigations about the effects and effectiveness of adaptivity for students with different learning styles. *2009 Ninth IEEE international conference on advanced learning technologies*, 415-419.

Graf, S., & Ives, C. (2010). A flexible mechanism for providing adaptivity based on learning styles in learning management systems. *2010 10th IEEE International Conference on Advanced Learning Technologies*, 30-34.

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A "proceptual" view of simple arithmetic. *Journal for research in Mathematics Education*, 116-140.

Gruber, T. R. (1993). A translation approach to portable ontology specifications. Knowledge acquisition, 5(2), 199-220.

Harel, G. (2018). The Learning and Teaching of Linear Algebra Through the Lenses of Intellectual Need and Epistemological Justification and Their Constituents. In *Challenges and Strategies in Teaching Linear Algebra* (pp. 3-27). Cham: Springer.

Hernández Díaz, A. et al. (2018). El método de análisis teórico de la actividad: una alternativa para el diseño curricular. *Dilemas Contemporáneos: Educación, Política y Valores, 6*(1).

Herrera Rojas, A. N. (2003) *Algunas consideraciones técnicas sobre la construcción de ítems de pruebas objetivas según la clasificación de objetivos educativos de Bloom.* Bogotá: Universidad Nacional de Colombia.

Herrera Rojas, A. N. (1998) *Notas sobre psicometría*. Bogotá: Universidad Nacional de Colombia.

Hibbs, A. D., Soussou, W. V., Jolly, J. D., & Fridman, I. V. (2018). U.S. Patent No. 10,068,490. Washington, DC: U.S. Patent and Trademark Office.

Ikwumelu, S. N. (2015). Adaptive Teaching: An Invaluable Pedagogic Practice in Social Studies Education. *Journal of Education and Practice, 6*(33), 140-144.

Kampwirth, T. J., & Bates, M. (1980). Modality preference and teaching method: A review of the research. *Academic Therapy*, *15*(5), 597-605.

Kaplan, R. M. & Saccuzzo, D. P. (2006) *Pruebas psicológicas. Principios, aplicaciones y temas.* México: Thomson

Karagiannis, I., & Satratzemi, M. (2016). A Framework to Enhance Adaptivity in Moodle. In *European Conference on Technology Enhanced Learning* (pp. 517-520). Cham: Springer.

Keefe, J. W. (1988). Profiling and Utilizing Learning Style. *NASSP Learning Style Series.*

Kirschner, P. A. (2017). Stop propagating the learning styles myth. *Computers & Education*(106), 166-171.

Kobal, D. (2018). Linear Algebra - A Companion of Advancement in Mathematical Comprehension . In *Challenges and Strategies in Teaching Linear Algebra* (pp. 279-298). Cham: Springer.

Kolb, D. (1984). *Experiential Learning: Experience as the Source of Learning and Development* . New Jersey: Prentice-Hall.

Kostolányová, K., & Šarmanová, J. (2014). Use of Adaptive Study Material in Education in E-Learning Environment. *Electronic Journal of e-Learning, 12*(2), 172-182.

Lapp, D. A. (2010). Student connections of linear algebra concepts: an analysis of concept maps. *International Journal of Mathematical Education in Science and Technology, 41*(1), 1-18.

Lee, J., & Park, O. (2008). Adaptive instructional systems. In e. a. J. M Spector, *Handbook of research on educational communications and technology* (pp. 469-484). New York: Lawrence Erlba.

Li, L. X., & Abdul Rahman, S. S. (2018). Students' learning style detection using tree augmented naive Bayes. *Royal Society open science, 5*(7), 172108.

Li, H. et al. (2018). Adaptive Learning System and Its Promise on Improving Student Learning. *Proceedings of the 10th International Conference on Computer Supported Education (CSEDU 2018), 2*, 45-52 .

Liu, Q. et al. (2019). Exploiting Cognitive Structure for Adaptive Learning. *arXiv preprint arXiv:1905.12470*.

Machado, M. et al. (2018). Uma Abordagem Evolutiva para o Problema de Sequenciamento Curricular Adaptativo. *Brazilian Symposium on Computers in Education (Simpósio Brasileiro de Informática na Educação-SBIE), 29*(1), 1243.

Magnusson, D. (1969). *Teoría de los Test*. México: Trillas.

Maravanyika, M. et al. (2017). An adaptive recommender-system based framework for personalised teaching and learning on e-learning platforms. *2017 IST-Africa Week Conference (IST-Africa)*, 1-9.

Martins, C. et al. (2008). An adaptive educational system for higher education. *Proceedings of the 14th EUNIS, 8*, 24-27.

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking Mathematically*. London: Addison Wesley.

Messick, S. (1989). Validity. En: R. L. Linn (Ed.), Educational Measurement (3rd ed., pp. 13- 104). New York: Macmillan.

Messick, S. (1995) *Validity of psychological assessment: Validation of inferences from persons' responses and performances as scientific inquiry into score meaning*. American Psychologist, 50(9), 741-749.

Moreno, J. et al. (2014). Una plataforma para la implementación de cursos en línea adaptativos: descripción y punto de vista de los docentes. *Revista Electrónica de Investigación Educativa, 16*(3), 103-117.

Mudrák, M. (2018). Personalized e-course implementation in university environment. *International Journal of Information and Communication Technologies in Education, 7*(2), 17-29.

Muñiz, J. y Hambleton, R. K. (2000). *Adaptación de los tests de unas culturas a otras.* Metodología de las Ciencias del Comportamiento, 2, 129-149.

Murray, M. C., & Pérez, J. (2015). Informing and Performing: A Study Comparing Adaptive Learning to Traditional Learning. *Informing Science: The International Journal of an Emerging Transdicipline* (18), 111.

Nafea, S. M. (2018). Ulearn: personalised learner's profile based on dynamic learning style questionnaire. *Proceedings of SAI Intelligent Systems Conference* (1105-1124).

Newman, A. et al. (2013). *Learning to adapt: Understanding the adaptive learning supplier landscape.* Education Growth Advisors.

Niss, M. A., & Højgaard, T. (Eds.) (2011). *Competencies and Mathematical Learning: Ideas and inspiration for the development of mathematics teaching and learning in Denmark.* Roskilde: Roskilde Universitet. IMFUFA-tekst: i, om og med matematik og fysik, N.

Oancea, R. et al. (2018). Adaptivity in E-Learning Systems. *International conference KNOWLEDGE-BASED ORGANIZATION, 24*(3), 66-69.

O'dwyer, B., & Krishnan, R. (2018). U.S. Patent Application No. 15/748,054.

Oktaç, A., & Trigueros, M. (2010). ¿Cómo se aprenden los conceptos de álgebra lineal? *Revista Latinoamericana de Investigación en Matemática Educativa, RELIME, 13*(4), 373-385.

Pedroza Flores, R. (2018). La universidad 4.0 con currículo inteligente 1.0 en la cuarta revolución industrial. *Revista Iberoamericana para la Investigación y el Desarrollo Educativo, 9*(17), 168-194.

Pérez, T., Gutiérrez, J., López, R., Gonzalez, A., & Vadillo, J. (2001). Hipermedia, Adaptación, Constructivismo e Instructivismo. *Revista Iberoamericana de Inteligencia Artificial*, 5(12), 29-38.

Pieters, J. V., & Pareja Roblin, N. (2019). *Collaborative curriculum design for sustainable innovation and teacher learning.* Springer Nature.

Pham, Q. D., & Florea, A. M. (2013). A method for detection of learning styles in learning management systems. *UPB Scientific Bulletin, Series C: Electrical Engineering, 75*(4), 3-12.

Polya, G. (1963). On learning, teaching, and learning teaching. *The American Mathematical Monthly*, *70*(6), 605-619.

Popsecu, E. et al. (2010). Accommodating learning styles in an adaptive education system. *Informatica* (34), 451-462.

Rasheed, F., & Wahid, A. (2019). Learning Style Recognition: A Neural Network Approach. *First International Conference on Artificial Intelligence and Cognitive Computing*, 301-312.

Resnick, L. B. (1973). Task analysis in curriculum design: a hierarchically sequenced introductory mathematics curriculum. *Journal of Applied Behavior Analysis, 6*(4), 679-709.

Rozo, H., & Real, M. (2019). Pedagogical Guidelines for the Creation of Adaptive Digital Educational Resources: A Review of the Literature. *Journal of Technology and Science Education, 9*(3), 308-325.

Scheiner, T., & Pinto, M. M. (2019). Emerging perspectives in mathematical cognition: contextualizing, complementizing, and complexifying. *Educational Studies in Mathematics*, *101*(3), 357-372.

Schwab, K. (2017). The fourth industrial revolution. Currency.

Sierpinska, A. (2000). On some aspects of students' thinking in linear algebra. In *On the teaching of linear algebra* (pp. 209-246). Dordrecht: Springer.

Skemp, R. R. (1986). *The psychology of learning mathematics.* Harmondsworth, U.K.: Penguin.

Snow, R. E. (1980). Aptitude, learner control, and adaptive instruction. *Educational Psychologist, 15,* 151–158.

Stewart, S., & Thomas, M. O. (2009). A framework for mathematical thinking: The case of linear algebra. *International Journal of Mathematical Education in Science and Technology, 40*(7), 951-961.

Stewart, S. (2018). Moving between the embodied, symbolic and formal worlds of mathematical thinking with specific linear algebra tasks. In *Challenges and strategies in teaching linear algebra* (pp. 51-67). Cham: Springer.

Stewart, S. (2018). *Challenges and strategies in teaching linear algebra.* Springer.

Taba, H. (1974). *Elaboración del currículo.* Buenos Aires: Editorial Troquel.

Tall, D. (1991). *Advanced mathematical thinking.* Springer Science & Business Media.

Tall, D. (1994). Understanding the processes of advanced mathematical thinking. *International Congress of Mathematicians*.

Tall, D. (2013). *How humans learn to think mathematically: Exploring the three worlds of mathematics.* Cambridge University Press.

Thorndike, R. L. (1995) *Psicometría aplicada*. México: Limusa.

Trigueros, M. (2018). Learning linear algebra using models and conceptual activities. In *Challenges and strategies in teaching linear algebra* (pp. 29-50). Cham: Springer.

Tyler, R. W. (1971). *Basic Principles of Curriculum and Instruction.* Chicago and London: The University of Chicago Press.

Union européenne. Commission européenne, & High Level Group on the modernisation of higher education. (2014). High Level Group on the Modernisation of Higher Education: Report to the European Commission on New Modes of Learning and Teaching in Higher Education. Publications Office of the European Union.

U.S. Department of Education, Office of Educational Technology. (2018, Abril 14). *Reimagining the role of technology in education: 2017 national education technology plan update.* Retrieved from https://tech.ed.gov/files/2017/01/NETP17.pdf

Vesin, B. et al. (2018). Learning in smart environments: user-centered design and analytics of an adaptive learning system. *Smart Learning Environments, 5*(1), 24.

Visscher-Voerman, J. I., & Gustafson, K. (2004). Paradigms in the theory and practice of education and training design. *Educational Technology, Research & Development, 52*(2), 69–89.

Weller, K., Montgomery, A., Clark, J., Cottrill, J., Trigueros, M., Arnon, I., & Dubinsky, E. (2002). Learning linear algebra with ISETL. *Recuperado de http://homepages. ohiodominican. edu/~ cottrilj/datastore/linear-alg/LLAWI-P3. pdf.*

Xie, H. et al. (2019). Trends and development in technology-enhanced adaptive/personalized learning: A systematic review of journal publications from 2007 to 2017. *Computers & Education.*

Yarandi, M., Tawil, A. R., & Jahankhani, H. (2012). Ontologies for personalised adaptive learning. In Advances in Computing & Technology. University of East London, School of Architecture Computing and Engineering.

APPENDICES

APPENDIX 1. Examples of challenging problems.

The proposal to students of challenging problems oriented to the development of mathematical thinking is a fundamental component of the proposed curriculum. This appendix shows examples of each of the types of challenging problems that were identified from the inductive analysis of the problems of the Colombian Mathematics Olympiads.

- **Problems where common calculations are needed, but for special objects or constraints.**

Problema 3. *Let A, B, C and D be vectors in A, B, C and D.*

1. *Prove that the set* $\{A+B, B+C, C+D, D+A\}$ *is linearly dependent.*
2. *If the set* $\{A, B, C, D\}$ *is linearly independent, shows that the set* $\{A+B+C, B+C+D, A+2D\}$ *is linearly independent.*

Figure 21. Problem 3 of node 3 arithmetic mode of thinking.
Author's own elaboration.

In this problem we are asked to determine the linear dependence or independence of a set of vectors, which is a common calculation, however, the proposed sets of vectors have special objects, since they are given by operations with vectors.

- **Problems where it is necessary to find objects with special characteristics.**

Problema 2. *If* $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, *draws all vectors* $B = \begin{bmatrix} x \\ y \end{bmatrix}$ *in the xy plane such that* $A \cdot B = 5$. *Which of all these B vectors is the shortest?*

Figure 22. Problem 2 of node 2 geometric mode of thinking.
Author's own elaboration.

In this problem, the students are asked to find the shortest vector that also satisfies the characteristic determined by the dot product. This also highlights the idea that all the vectors that satisfy a condition must be drawn, although the number of vectors that satisfy the condition is infinite.

- **Problems where the question is reversed, i.e., from a solution the student is asked to propose a question.**

Problema 5. *Use the definition of the inner product presented in the previous exercise and define the norm of a vector and a unit vector in* $\mathbb{R}^2$.

Figure 23. Problem 5 of node 4 structural mode of thinking.
Author's own elaboration.

Normally, students are provided with the definitions, and they apply them. In this problem the question is the opposite: they must define the norm of a vector and a unit vector in the two-dimensional space from the consideration of an exercise done earlier.

- **Problems where examples or counterexamples are requested for certain mathematical results.**

Problema 1. *Solve this problem with two other partners. Each must do a part.*

Let $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ -6 \\ 5 \end{bmatrix}$ *y* $C = \begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix}$ *vectors in* $\mathbb{R}^3$.

Part 1: Verify that A and B are orthogonal.
Part 2: Verify that B and C are orthogonal.
Part 3: Verify that A and C are not orthogonal.
Discuss among the three of you the reasons for this behavior. Make an illustration of this situation.
Is it possible to find a vector C' *that is orthogonal to A and B?*
If it is possible, each team member should show an example.

Figure 24. Problem 1 of node 2 arithmetic mode of thinking.
Author's own elaboration.

In this problem, students must provide examples of a vector that satisfies the condition of being orthogonal to the given A and B vectors. This problem involves students transcending the two-dimensional thought form and placing themselves in three-dimensional space. It is emphasized here that the problem proposes a discussion among the team members and that, although the problem is situated in the arithmetic mode of thinking, they are asked to make a change of representation and make graphical illustrations of the situation.

- **Problems asking for full or partial proofs of mathematical results.**

Problema 11. *Solve this problem with two other partners. Each one must do two non-consecutive parts.*
Part 1. Invent two distinct unit vectors in $\mathbb{R}^2$.
Part 2: Calculate the angle between these two vectors.
Part 3: Check that the vector resulting from the sum of these two vectors bisects the angle between them.
Part 4: Give an example to show that it is not always the case that the sum vector bisects the angle between two given vectors.
Part 5: Gives a necessary condition that two vectors must satisfy in order for the sum vector to bisect the angle between them.
Part 6: Is it necessary to know the measure of the angle between the two vectors in order to show that the sum vector bisects the angle?

Figure 25. Problem 11 of node 1 geometric mode of thinking.
Author's own elaboration.

Although the general criterion was not to propose demonstrations of mathematical propositions to students as challenging problems, in some cases, as in the problem above, 'hidden' demonstrations were proposed, in which students were given the steps to reach a proposition and at the end they were asked to draw conclusions from the process.

We also proposed problems that are not marked in the previous classification but were obtained from inspiration in books and other didactic resources.

APPENDIX 2. Genetic decomposition of the content to be taught for each of the modes of thinking.

The following is a genetic decomposition, framed in the synthetic-geometric mode of thinking, for the content that will be used as a pretext in this module for the development of mathematical thinking.

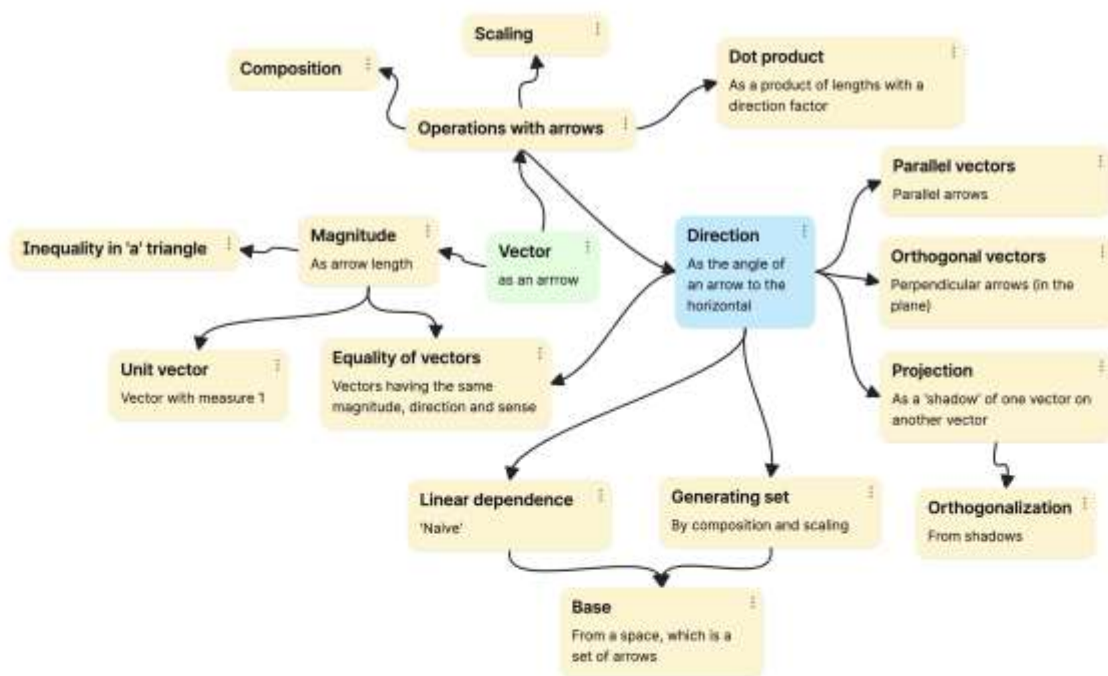


Figure 26. A genetic decomposition of the content to be taught in the module (synthetic-geometric mode of thinking).

Author's own elaboration.

The previous genetic decomposition is centered on the concept of vector, conceived as an arrow, so the other concepts are related to that conception. The centrality of the concept of direction, which is not obvious in the first instance, is noted.

The following is a genetic decomposition, framed in the analytic-aritmetic mode of thinking, for the content that will be used as a pretext in this module for the development of mathematical thinking.

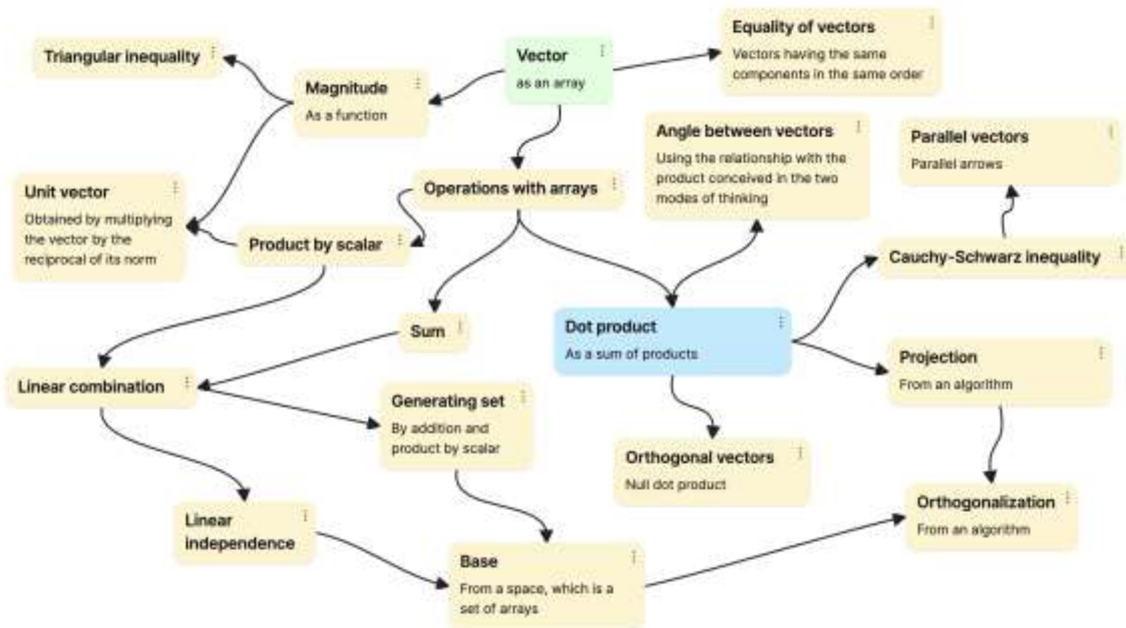


Figure 27. A genetic decomposition of the content to be taught in the module (analytic-arithmetic mode of thinking). Author's own elaboration.

Of course, this genetic decomposition is also centered on the concept of vector, but the protagonism is obtained in this one by the dot product.

Note: it is important to emphasize that none of the above is 'the' genetic decomposition of this construct, but it is one of the possible genetic decomposition options.

APPENDIX 3. Instructions for students in each of the two iterations.

In each of the two interactions, instructions were provided to the students on how to use the ALMS in their learning process. instructions to the students so that they could learn how to use the ALMS in their learning process. The two instruction documents are presented below.

Instrucciones para el uso del sistema de aprendizaje adaptativo

El sistema de aprendizaje adaptativo en el que te apoyarás para alcanzar los resultados de aprendizaje esperados para este módulo te permitirá seguir una secuencia de actividades acordes al modo de pensamiento que tienes con respecto a los objetos matemáticos del álgebra lineal.

Para lograr esto, inicialmente el sistema te propondrá una prueba de caracterización de modos de pensamiento. Resuélvela lo más honestamente posible, porque de ello dependerá tu comodidad en el avance en el proceso de aprendizaje.

Aparte de la plataforma del sistema de aprendizaje adaptativo, te apoyarás también en el chat del equipo de Teams. Deberás usarlo en algunos momentos específicos del proceso de aprendizaje, pero también puedes usarlo si tienes alguna duda en cualquier otro momento **(dentro del espacio de la clase o el tiempo de tutorías)**.

El módulo que estás por empezar está constituido por cuatro nodos:

1. Fundamentos sobre vectores
2. Producto escalar y proyecciones
3. Ortogonalización
4. Espacios vectoriales

Para usar el sistema de aprendizaje adaptativo:

1. Ingresa a https://adaptive-elearnign-system.web.app/
2. Regístrate usando tu cuenta de correo institucional.
3. Selecciona el grupo de la asignatura que te corresponde en este semestre.
4. Resuelve la prueba de caracterización de modos de pensamiento.
5. Ingresa al primer nodo de aprendizaje (fundamentos sobre vectores).
6. Desarrolla cada uno de los momentos de ese nodo. Se llaman momentos a cada una de las tarjetas que te presentará el sistema. Hay diferentes tipos de momentos:
   a. Momento para la presentación del nodo.
   b. Momentos para el desarrollo teórico del nodo.
   c. Momentos en los que se te harán preguntas para verificar tu comprensión.
   d. Momentos en los que deberás resolver ejercicios, para que ganes comprensión y destreza en los algoritmos.
   e. Momentos en los que deberás resolver problemas, para que desarrolles tu pensamiento matemático.
   f. Momento con el mapa del nodo.
7. Cuando llegues a algún momento que contiene un problema, resuélvelo. Podrás trabajar los problemas en la clase o fuera de esta, sin embargo, deberás discutir la solución de cada uno de los problemas con tu profesor, y eso sólo podrás hacerlo en la clase. Cuando vayas a discutir las soluciones de los problemas con tu profesor en clase, por favor escríbele en el chat el número del problema en el que vas y el modo de pensamiento en el cual quedaste clasificado.
8. En algunos problemas tendrás que trabajar con tus compañeros. Podrás hacer uso del chat para encontrar compañeros que vayan en el mismo problema que tú. (Recuerda que debes buscar compañeros que sean de tu mismo modo de pensamiento).
9. Los tiempos estimados para el desarrollo de cada uno de los nodos, es el siguiente:

Hito	Clases
Prueba de caracterización	0
Nodo 1: Introducción a vectores	4
Nodo 2: Producto escalar y proyecciones	3
Nodo 3: Ortogonalización	2
Nodo 4: Espacios vectoriales	2

10. Cuando llegues al final del nodo, comunícate con tu profesor, quien determinará si puedes avanzar al siguiente nodo o deberás cambiar de nodo de acuerdo con tu modo de pensamiento.
11. Si el profesor determina que debes cambiar de modo de pensamiento, cierra la sesión y vuelve a hacer *login*.

Figure 28. Instructions to students in the first iteration.

Author's own elaboration.

INSTRUCCIONES PARA EL USO DEL SISTEMA ADAPTATIVO

Para usar el sistema de aprendizaje adaptativo para desarrollar el módulo de álgebra lineal de la asignatura solución de problemas, por favor sigan estas instrucciones:

1. Desde un computador o Tablet, ingresen a https://adaptive-elearnign-system.web.app/. No usen celular para navegar en este sistema.
2. Con base en la prueba de caracterización de modo de pensamiento que contestaron, serán incluido(a)s en un grupo de trabajo que tendrá un líder. Desarrollarán este módulo de la asignatura trabajando en conjunto con los compañero(a)s de su grupo.
3. Cada grupo de trabajo contará con un grupo de WhatsApp que facilitará la comunicación y por el cual podrán hacer preguntas y recibir respuestas cuando estén estudiando.
4. El módulo de álgebra lineal está compuesto de cuatro nodos:
    a. Fundamentos sobre vectores
    b. Proyecciones
    c. Ortogonalización
    d. Espacios vectoriales

   Deben empezar por el primer nodo y cuando lo terminen, les será habilitado el siguiente nodo.
5. Cada nodo está compuesto de tarjetas, llamadas momentos de aprendizaje. Estos momentos son de diferentes tipos:
    a. Momento de presentación del nodo: allí encontrarán el objetivo y los contenidos del nodo. También podrán ver el número de problemas que deben resolver en cada nodo.
    b. Momentos de desarrollo teórico: contienen la teoría, que deberán leer comprensivamente para la resolución de los problemas.
    c. Momentos con preguntas de comprensión: resuelvan la o las preguntas que se les hacen allí. Después de estos momentos encontrarán la retroalimentación, con la que podrán verificar su comprensión sobre la teoría presentada. Pregunten al profesor si tienen dudas con respecto a las enseñanzas verificadas en este momento.
    d. Momentos con ejercicios: allí podrán ganar destreza sobre los algoritmos presentados en la teoría. Después de cada momento que contenga ejercicios, tendrán un momento con su retroalimentación para que comprueben sus resultados. Si tienen inquietudes en los ejercicios, pueden preguntar al profesor.
    e. Momentos con problemas: allí se les presentarán problemas, que deberán resolver en el cuaderno. Cuando hayan resuelto cada problema, deben avisar al profesor por Whatsapp para que él lo revise y les haga la retroalimentación, en los momentos de clase.
    f. Momento con mapa del nodo: allí encontrarán un resumen de lo aprendido durante el nodo.
6. Durante las sesiones de clase, trabajarán en la plataforma y se comunicarán a través de la sesión de GoogleMeet. Si durante el trabajo fuera de clase tienen dudas, el líder escribirá por WhatsApp al profesor para que se conecte y les ayude a resolverlas.
7. Una vez terminen cada nodo, el líder debe informar al profesor, quien les dará acceso al siguiente nodo.
8. La evaluación del módulo de álgebra lineal, que tiene un valor del 30% de la asignatura, se hará de la siguiente manera:
    a. 3% será dedicado a la autoevaluación del proceso de aprendizaje.
    b. El 27% restante se obtendrá del promedio de la calificación de cada uno de los cuatro nodos.
    c. La calificación de cada nodo se obtendrá haciendo el promedio del 50% de las mejores calificaciones de los problemas.

La distribución de tiempos para el trabajo en los nodos es la siguiente:

Nodo 1: 4 clases – 8 horas presenciales – 16 horas de trabajo independiente
Nodo 2: 2 clases – 4 horas presenciales – 8 horas de trabajo independiente
Nodo 3: 2 clases – 4 horas presenciales – 8 horas de trabajo independiente
Nodo 4: 2 clases – 4 horas presenciales – 8 horas de trabajo independiente

Recuerden que, según el sistema de créditos universitarios, deben disponer de trabajo extraclase para avanzar en el sistema adaptativo.

Figure 29. Instructions to students in the second iteration.

Author's own elaboration.

APPENDIX 4. Framework for test design to classify students into modes of thinking.

The purpose of this assessment is to make an approximation to the predominant mode of thinking in the students participating in this research (in the framework of Sierpinska's (2000) modes of thinking), to establish which line of learning linear algebra will be used.

Although existing psycho-technical tests use related concepts such as intelligence, aptitudes, problem solving or learning styles, they are structured on conceptual references that lead to general evaluations of these constructs, or to assessments of factors that are not of interest for the stated purpose.

Thus, for example, they point out that intelligence is manifested in performance in various tasks, as predictors of academic success and, therefore, they focus attention on the result or manifestation and not on the process used to arrive at that response, an approach like that made by Alfred Binet in 1905. For their part, evidence on learning strategies refers to "all those cognitive, affective, and motor procedures that students mobilize in a conscious and reflective manner, oriented towards the effective achievement of a specific learning goal or objective. To plan, control, regulate, and evaluate the incidence of the variables that influence their learning" (Ferreras, 2008, p.51).

Since it is a question of evidencing conditions from the synthetic or analytical processes, as ways of thinking, after a review of some of the existing instruments, the construction of an instrument is chosen.

Review of existing instruments

Table 9. Review of existing instruments for thought characterization.

Author's own elaboration.

Test	Description	Purpose	Disadvantage
Matrices	Estimation of general intelligence by means of a non-verbal abstract reasoning task based on graphic matrices. Each matrix consists of 9 elements (3 rows X 3 columns) to which a piece has been 'erased' to discover the logic that relates the figures to each other.	Evaluates the person's ability to: understand and establish relationships, abstract and perform processes of deduction and induction, reason and make judgments from different contents and information; establish sequences and relationships between elements, distinguish relevant and superficial characteristics; mentally perform different operations using intensively the working memory; compare information from two or more sources to reach conclusions.	Establishes intelligence grading. Abilities should be relatively independent of specific task content. Predicts performance on a uniquely structured curriculum.
Factor g-R	These are non-verbal tests. In order to perform them, it is required that the person perceives the possibility of relationship between abstract shapes and figures.	The g-R Factor is made up of 4 tests (Series, Classification, Matrices and Conditions) that should be applied together, although they use independent times and appreciate different aspects of intelligence.	It is a test of intelligence, but it is not related to the characterization of modes or styles of thinking.
BAT-7 Skills assessment	The battery consists of 3 levels or booklets of increasing difficulty suitable for different types of school children and adults.	Verbal Aptitude (V), Spatial Aptitude (E), Attention (A), Concentration (CON), Reasoning (R), Numerical Aptitude (N), Mechanical Aptitude (M) and Orthography (O). It provides a score on the attentional style of the person and allows estimating the main intellectual factors of the cognitive system: Factor g or general ability (g), Fluid Intelligence (Gf) and Crystallized Intelligence (Gc).	It is a test of intelligence, but it is not related to the characterization of modes or styles of thinking.
OTIS Simple. General Intelligence Test (b)	The test consists of a selection of 75 items that measure different aspects of intelligence.	It evaluates factors such as deductive and inductive reasoning and lexical knowledge.	Instrument designed to measure the intellectual capacity of subjects with a low or medium cultural level. It is used in personnel selection where the level

			of education of the candidates is low or medium.
D-48. Dominoes test (b)	Measurement of the "g" factor. Uses non-verbal stimuli, requires very little prior knowledge.	It evaluates central functions of intelligence such as abstraction, understanding of relationships, ability to conceptualize and apply systematic reasoning to new problems.	Widely used, both in personnel selection and in school evaluation, it offers a wide range of criteria, but it is not oriented to the characterization of modes of thinking.
D-70. Dominoes test (b)	Developed as a parallel version of D-48 and with a similar level of difficulty.	It is suitable for medium and higher levels and is very useful to replace the D-48 when it is known.	Same disadvantage as in the previous test.
CEA Learning Strategies Questionnaire	It is based on a model of mental functioning to study the different strategies that students can bring into play in the learning process.	It evaluates four scales or processes and 11 subscales, in which the strategies are grouped: 1. Awareness (motivation, attitudes, affectivity-emotional control). 2. Processing (selection, organization and elaboration of information). 3. Personalization (critical and creative thinking, retrieval, transfer). 4. Metacognition (planning, regulation).	School use The test makes it possible to identify the strategies employed by students and to make study recommendations. It characterizes aptitudes, but not modes of thinking.
ACRA. Learning Strategies (a)	Evaluates learning strategies in secondary school.	It is framed within the learning theory and establishes scales that evaluate the use that students habitually make of: 1. information acquisition, 2. information encoding, 3. information retrieval and 4. processing support.	High number of items (119). Writing of items that are not very comprehensible and intelligible.

Test design. Modes of thinking report.

A test is "an instrument or measurement technique used to quantify behavior or to help understand and predict behavior" (Kaplan & Saccuzzo, 2006, p. 6). An objective test is one in which the results obtained are independent of the person scoring the test (Herrera, 2003). In other words, the result in an objective test depends on the response given by the person being tested, that is, on his or her capacity, ability, knowledge, state or trait, as the case may be, and not on the appreciation of the person observing the behavior to be measured and evaluated.

The structured nature of tests is summarized by Herrera (2003) from the work of Brown (1980) and Thorndike (1995) as follows: "Objective tests are constructed by means of systematic and standardized procedures that allow the measurement of a construct or a property defined within a theoretical body of psychology and allow its quantification" (p.2). In this sense, a structured test responds to a general process of design, construction, validation, application, scoring and duly controlled analysis, which delimits the largest possible number of variables that can explain the results obtained by a person in the execution of a rigorously predefined activity.

Objective tests with a structured base are made up of instructions, items, record of answers and complementary material for construction, application, and interpretation. According to Herrera (1998), a specific procedure must be followed for the construction of a test, which includes:

1. Planning. It involves the definition of the following aspects (Aiken, 1996; Herrera, 1998): a. Object of measurement. b. Target population. c. Objective of the test. d. Conceptual framework. e. Test structure. Test structure.
2. Construction. This involves the design of the psychometric specifications and the elaboration of complementary material such as application and scoring instructions. It also involves the validation of the items and of the test.
3. Analysis and scoring of the test. The results of the test are reported.

1. Planning

Object. The object in this case is established from the approach of Sierpinska (2000) regarding the three modes of thinking in linear algebra. For characterization purposes, only the modes of thinking are used: synthetic-geometric and analytic-arithmetic. These two modes of thinking are analyzed on the basis of four factors, two of them oriented towards the geometric-arithmetic disjunction and two towards the synthetic-analytic disjunction.

Factors associated with the synthetic-analytic disjunction include:

- Action or pretension: refers to the ways in which people act in situations.
- Attitude: refers to the posture that people manifest when faced with situations.

Factors associated with geometric-arithmetic disjunction are as follows:

- Object relation: refers to the way people perceive objects to facilitate their understanding.
- Form of expression: refers to the form of language that people prefer to refer to objects.

Indicators for each of the modes of thinking in each factor are proposed below.

Table 10. Indicators for factors used in the characterization of modes of thinking.

Author's own elaboration.

Factors	Modes of thinking	
	Synthetic-Geometric	Analytical-Arithmetic
Relationship with the object	The object is given directly.	The object is given indirectly.
Action or claim	Concrete. Attempts to describe the object. Visualizes positions in space. Visualizes relationships between objects (vectors, lines, planes). Visualizes all possible cases (e.g., of lines in three-dimensional space). Graphically represents possible solutions to a system of equations.	Abstraction. Try to find possible solutions (forms of organization). Simplify calculations. Substitute variables. Use formulas. Solve systems of equations.
Attitude	Practical.	Theoretical.
Form of expression	Figure language, graphic representations. Direct.	Figures are understood as sets with conditions. Systemic.

a. Target population. Mathematics students enrolled in the linear algebra course.

b. Test purpose: This instrument is intended to make an approximation to the recognition of the predominant mode of thinking with which a student performs the learning of mathematical objects (according to Sierpinska's (2000) framework).

c. Test structure. This test is self-referencing and self-application, which implies that the student performs a referencing process on his preferences in the approach to mathematical objects, so that there are no adequate or inadequate answers.

A set of statements is used that are scored on a Likert scale:

1 indicates full agreement with the option in the left-hand column.
2 indicates agreement with the option in the left column most of the time.
3 indicates an intermediate answer (sometimes one and sometimes the other) between the option in the left column and the option in the right column.
4 indicates agreement most of the time with the option in the right-hand column.
5 indicates full agreement with the option in the right-hand column.

Statements are constructed according to the characteristics of each mode of thinking.

2. Construction

The construction of the items is developed according to the type and format of question chosen in accordance with the purpose. In ideal situations, psychometric analysis allows decisions on item quality to be made through test pilots and with definitive data from a sample of the target population. However, in those cases where it is necessary to control the prior encounter with the test content, these analyses are done a posteriori and decisions are made based on a priori established criteria.

Since the construction, application and interpretation of a test may involve variability for the person being evaluated, supplementary material must be developed to control errors at these moments. In terms of construction, errors associated with the form, relevance, pertinence, pertinence and difficulty of the items are controlled. Similarly, in the application, mechanisms and instructions are established to control the environment, time and social interactions in such a way as to guarantee equality for the examinees and a minimum interference of factors external to the individual characteristics in order to respond to the test.

Validation

Validity in psychometrics is understood as the fact that a test measures what it claims to measure (Magnunsson, 1969; Aiken, 1996; Herrera, 1998; Muñiz & Hambleton, 2000; Kaplan & Saccuzzo, 2006). Validity can also be understood as the level of empirical and theoretical support for the interpretations of the scores obtained in a test according to its object of measurement (Kane, 2001).

Messick (1989, 1995) suggests the following as purposes of validity:

- Analyze the relevance and representativeness of the test contents.
- Determine ethical and social consequences derived from the existence of bias in the tests.
- Define the structure of the test in relation to its conformation and dimensions.
- To identify the usefulness of measurement based on the study of the relationship between tests.
- To account for the theoretical consistency of the observed responses.
- To generalize the results obtained to other populations, situations or activities, which explains the translation and adaptation of tests.

The construction of items is delegated to experts in each of the topics or areas covered by the test and each of these items (reagent or statement), before being presented to the examinee, must undergo a validation process.

The apparent and content validity is analyzed.

All items must be approved in relation to the following content validity criteria:

- pertinence (the item fits the purposes of the test),
- relevance (for the purposes of the test it is important to ask that question),
- difficulty (subjective assessment of the degree of skill required to answer the item), and

- grammatical neatness (the item adequately uses the rules of language usage).

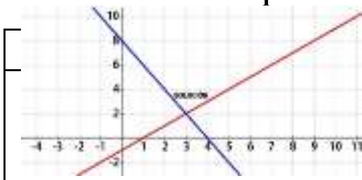
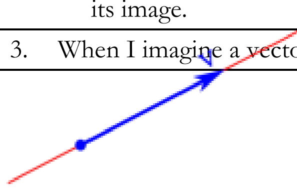
Every item must pass an evaluation associated with face validity, that is, that it gives the impression of measuring what it claims to measure or for the purpose for which it is intended, for which statistical methods are used, which in this case are not relevant and are substituted through the evaluation of judges.

Table 11. Types of validations for the mode of thinking characterization test.
Author's own elaboration.

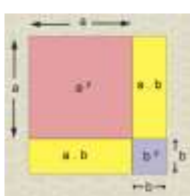
Type of validity	Description	Calculation procedure	Criteria for decision making
Apparent validity	The impression that an item and a test measure the construct it claims to measure. This criterion is fundamental in selection processes because the people who compete usually demand that the questions be oriented to their future performance contexts.	Peer rating on a scale of 0 to 3, where 0 is the undesired value and 3 is the desired value.	All tests and items that appear to measure what they claim to measure should be accepted.
Content validity	The items assess the construct they claim to measure. In this case, it goes beyond appearance; it is a conceptual correspondence between the construct and each question of the test. Due to the characteristics of this type of validity, there are no numerical criteria for its calculation, so the following process is used as an integral part of the validation (Martínez, 1996; Muñiz, 1996; Herrera 1998; McGartland, Berg-Weger, Tebb, Lee, & Rauch, 2003): 1. Framework of the initial specifications of the test. 2. Selection of the valid items within the possible universe. 3. Differentiation and descriptors of the given construct in the subject. 4. Identification of persons with expertise in the construct. 5. Conduction of item construction workshops.	Index of agreement between observers or judges. The evaluation criteria are pertinence, relevance, grammatical mastery. Peer assessment on a scale of 0 to 3, where 0 is absence of the criterion and 3 is desired presence of the criterion.	In test construction workshops the entire team of experts must accept the item on all criteria.

Design of the questions (classified by factors)

Factor: Relationship with the object

…n of equations, I prefer to see its solution as follows:					
	1	2	3	4	5	$x = 3$ $y = 2$
2. I prefer to know an object by						
its image.	1	2	3	4	5	its features.
3. When I imagine a vector, I think of						
	1	2	3	4	5	$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
4. When faced with the need to describe a place, I prefer to						
make the scheme or drawing.	1	2	3	4	5	list their characteristics.

Factor: Form of expression

5. To assemble a piece of furniture, I prefer to be guided by						
images of what to do.	1	2	3	4	5	instructions in the manual (no images).
6. To understand that $(a + b)^2 = a^2 + 2ab + b^2$, I prefer						
	1	2	3	4	5	$(a + b)(a + b)$ $= aa + ab + ba + bb$ $= a^2 + 2ab + b^2$
7. I agree more with the phrase:						
"The whole is the sum of the parts"	1	2	3	4	5	"Each part is a whole"
8. In order to know the characteristics of a conic section, I prefer to use its						
graphic.	1	2	3	4	5	equation.

Factor: Action or pretensión

9. When I have to get somewhere new, I prefer to find my way around with						
a map.	1	2	3	4	5	an address.
10. "What is the number that when added to its double gives 24?". To solve this problem, I prefer						
try out the solution.	1	2	3	4	5	formulate equations.
11. When giving directions to a friend to my home, I prefer to						
locate important points of reference.	1	2	3	4	5	set up a route with turns, paths and times.
12. I identify more with someone who builds a construction.						
by intuition.	1	2	3	4	5	based on calculations.

Factor: Attitude

13. To know a story, I prefer						
watch a movie.	1	2	3	4	5	read a book.
14. When setting up a new electronic device, I prefer to						
interact with it.	1	2	3	4	5	read the manual.
15. When faced with a problem, I prefer						
try several possible solutions.	1	2	3	4	5	consult the theory to find the solution.
16. When I have to make a decision, I am guided more by						
my intuitions.	1	2	3	4	5	the judgment of the facts.